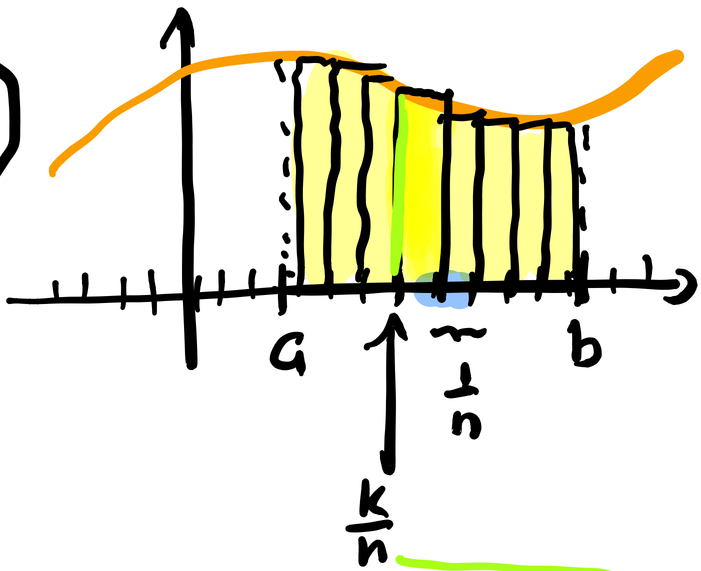
Unit 15

Double integrals

(1) Riemann sums

(1D)



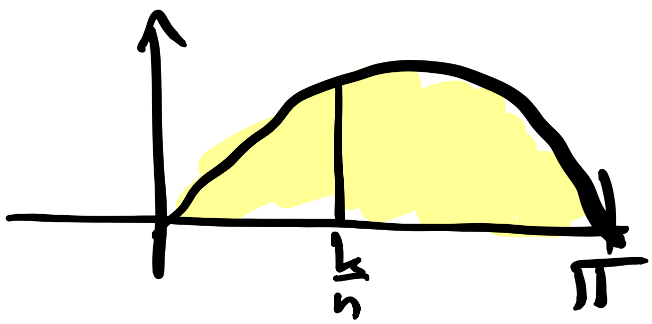
Archimedes

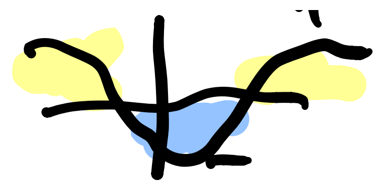
Riemann sum

$$\sum_{a \le \frac{k}{n} < b} \boxed{f\left(\frac{k}{n}\right)} \boxed{\frac{1}{n}}$$

$\downarrow \; n \to \infty$

Riemann integral $\boxed{\int_a^b f(x)\,dx}$





you are doing the sum

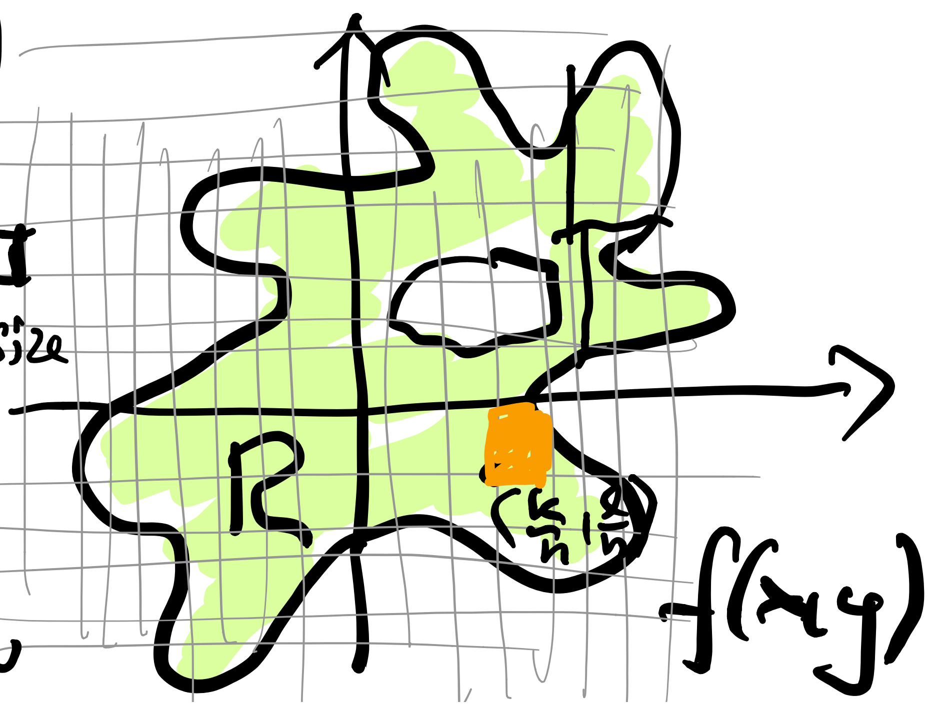
$$\sum \sin\left(\frac{k}{n}\right) \frac{1}{n}$$
$$0<\frac{k}{n}<\pi$$

$$\int_0^\pi \sin x \, dx = -\cos x \Big|_0^\pi = \boxed{2}$$

2D

$\frac{1}{n}$ grid size

n large number



$$\sum_{\left(\frac{k}{n},\frac{l}{n}\right) \text{ in } R} f\left(\frac{k}{n},\frac{l}{n}\right)\frac{1}{n^2}$$

$$\downarrow n\to\infty$$

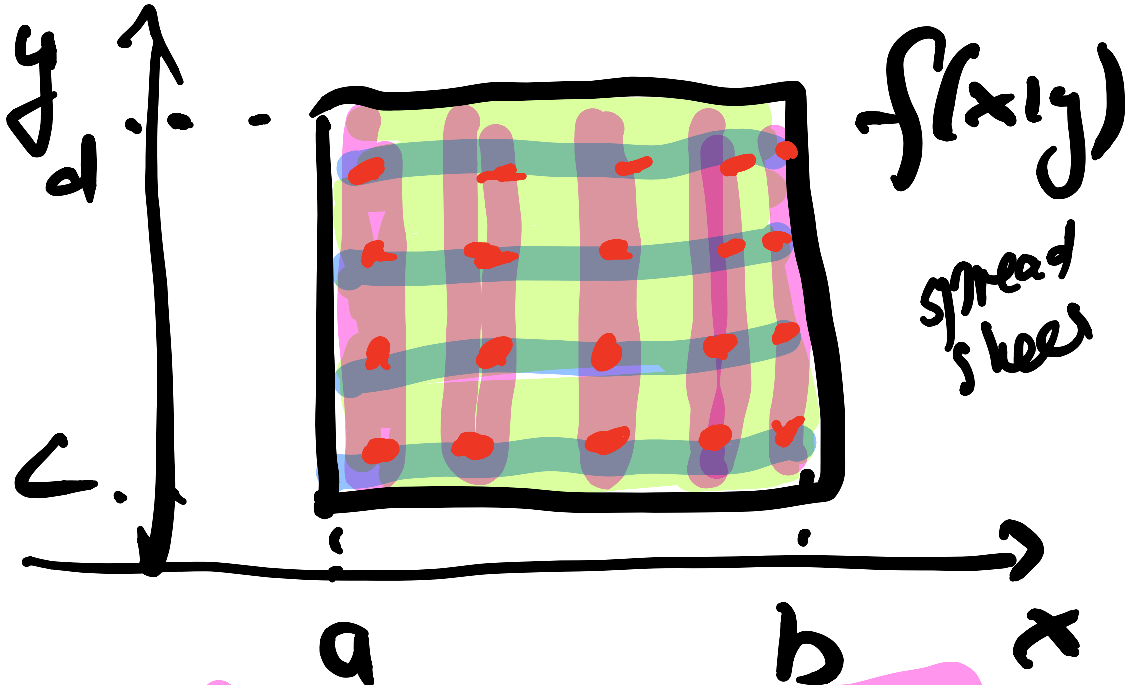
Integral in 2D

$$\boxed{\iint_R f(x,y)\,dA}$$

Geometric interpret:

Signed volume above R below the graph of f

② Fubini theorem



$$\sum_{a \le \frac{k}{n} < b} \boxed{\sum_{c \le \frac{l}{n} < d} f\left(\frac{k}{n}, \frac{l}{n}\right) \frac{1}{n}} \frac{1}{n}$$

$$\downarrow \ n \to \infty$$

$$\int_a^b \boxed{\int_c^d f(x,y)\, dy}\, dx$$

Reduction to 1D integration

$$\int_c^d \boxed{\int_a^b f(x,y)\,dx}\,dy$$

$$\uparrow$$

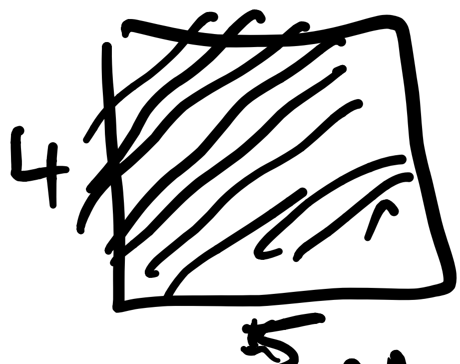
$$\sum_{c \le \frac{\ell}{n} < d} \boxed{\sum_{a \le \frac{k}{n} < b} f\left(\frac{k}{n}, \frac{\ell}{n}\right)\frac{1}{n}}\frac{1}{n}$$

Fubini's theorem

$$\boxed{\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy}$$

Wittgenstein:

$$4 \times 5 = 5 \times 4$$



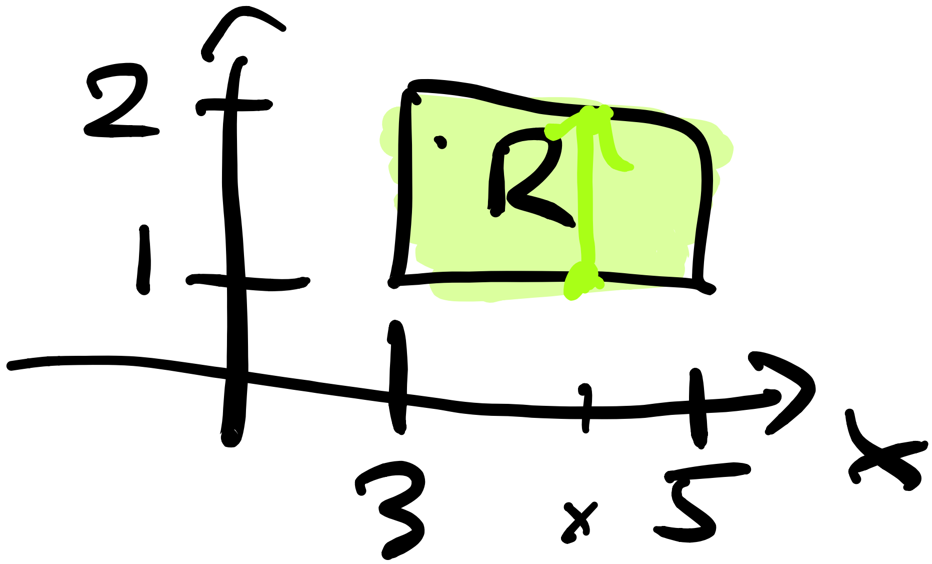
"commuting"

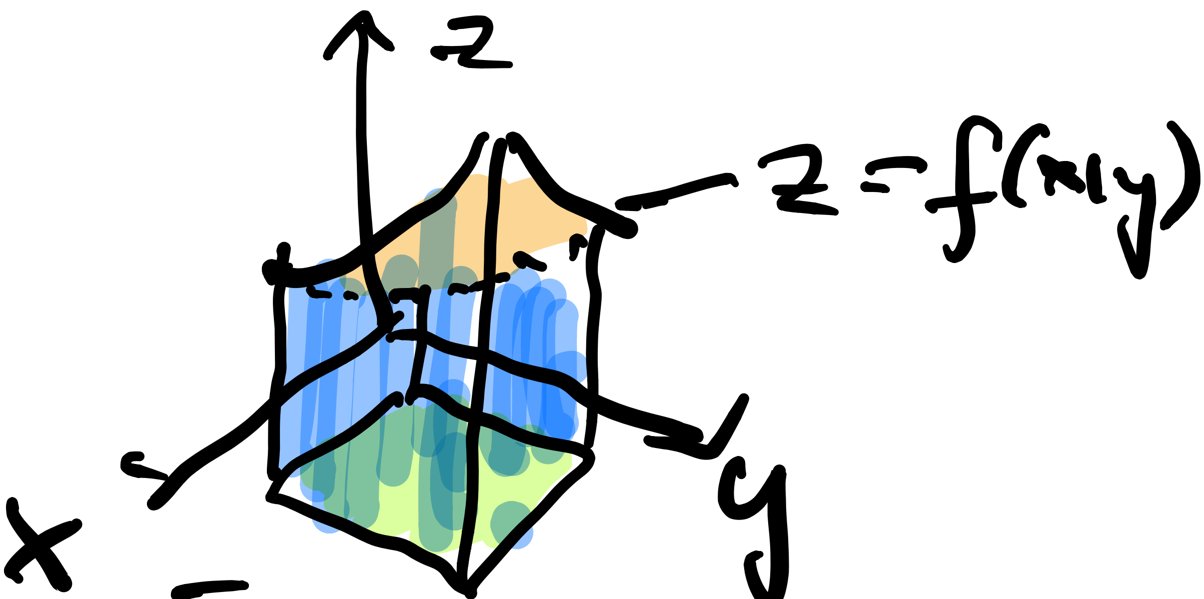
same area ..

In reality this is not true! (QM)

$$pq - qp = \hbar$$

$$\iint_R x^2 y \, dA$$





$$\int_3^5 \int_1^2 x^2 y \; dy\, dx$$

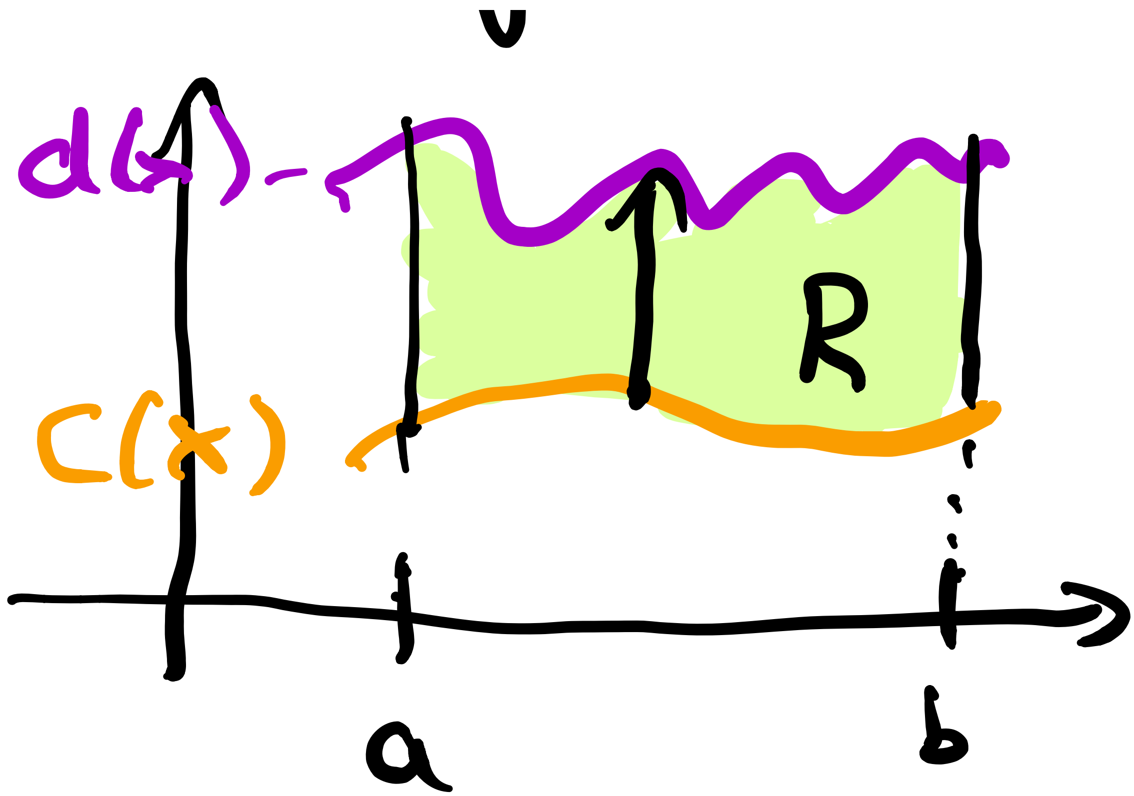
$$\int_3 \int_1 \frac{x\,y}{2}\Big|_{1=y} = \frac{3}{2}x^2$$

$$\frac{\cancel{3}}{2}\,\frac{x^3}{\cancel{3}}\Big|_3^5 = \frac{125 - 27}{2} = \frac{98}{2}$$

$$= 49$$

$$\int_1^2 \int_3^5 x^2 y \, dx\, dy$$

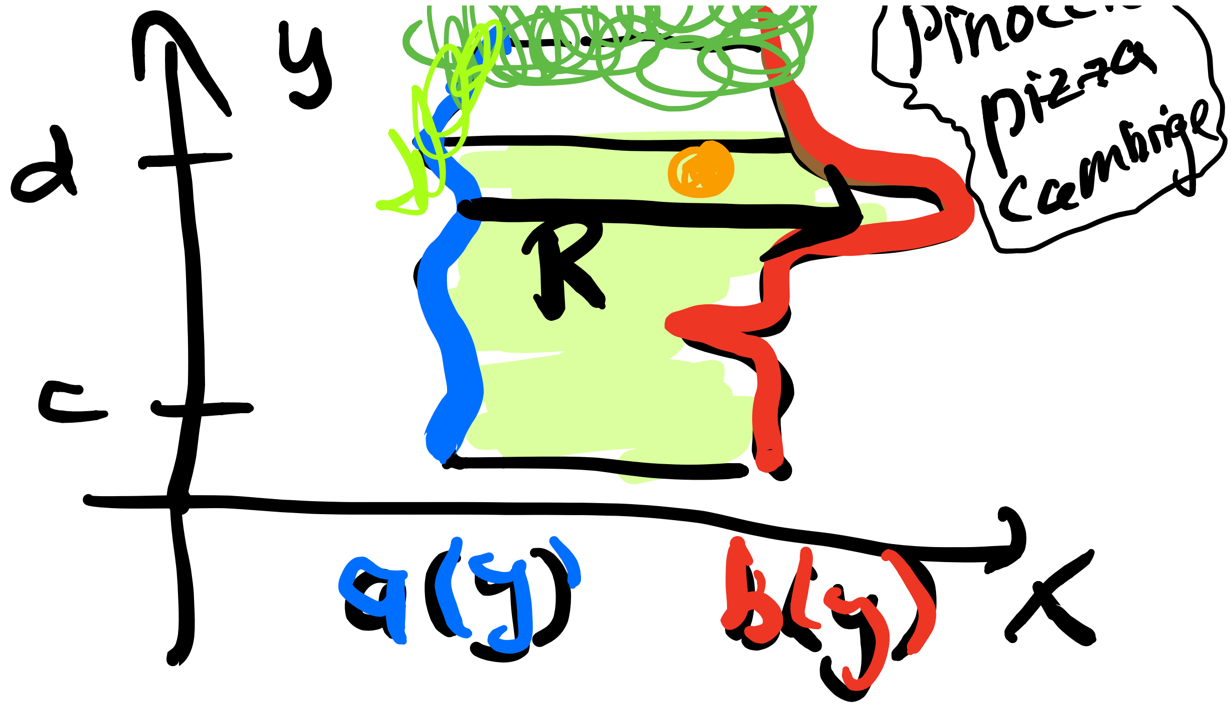
③ More general regions



bottom to top integral

type I (Stewart)

$$\int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx$$



left to right

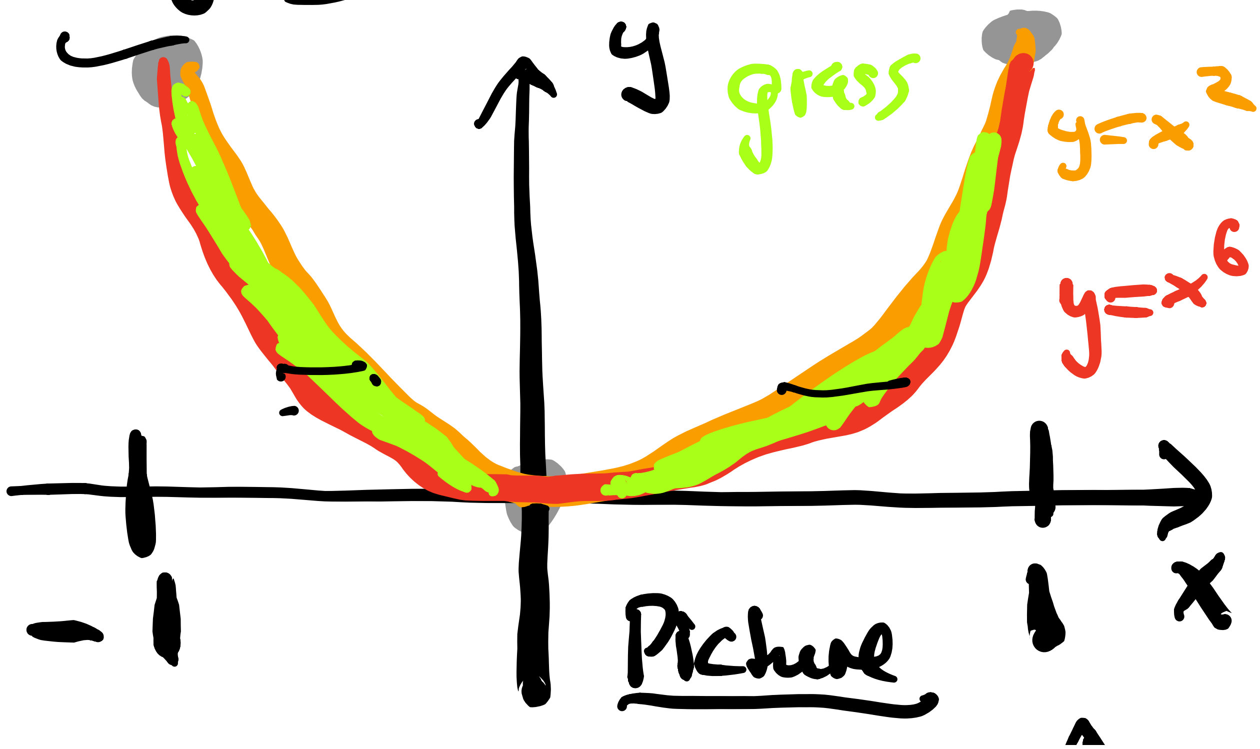
$$\int_{c}^{d} \int_{a(y)}^{b(y)} f(x,y)\,dx\,dy$$

(4) Examples

a) **Problem**

What is the integral of $f(x,y)$ over the region bound by $y=x^6$ and $y=x^2$

$f(x,y) = y$



Picture

Set up integral bottom to top

$$\int_{-1}^{1}\int_{x^6}^{x^2} y \; dy\, dx$$

$$\frac{x^5}{10} - \frac{x^{13}}{26}\Big|_{-1}^{1}$$

$$\frac{y^2}{2}\Big|_{x^6=y}^{x^2=y} = \frac{x^4}{2} - \frac{x^{12}}{2}$$

$$\int_{-1}^{1} \frac{x^4}{2} - \frac{x^{12}}{2}\, dx = \frac{1}{5} - \frac{1}{13}$$

$f(x)$

$$= \boxed{\frac{8}{65}} = \ldots$$

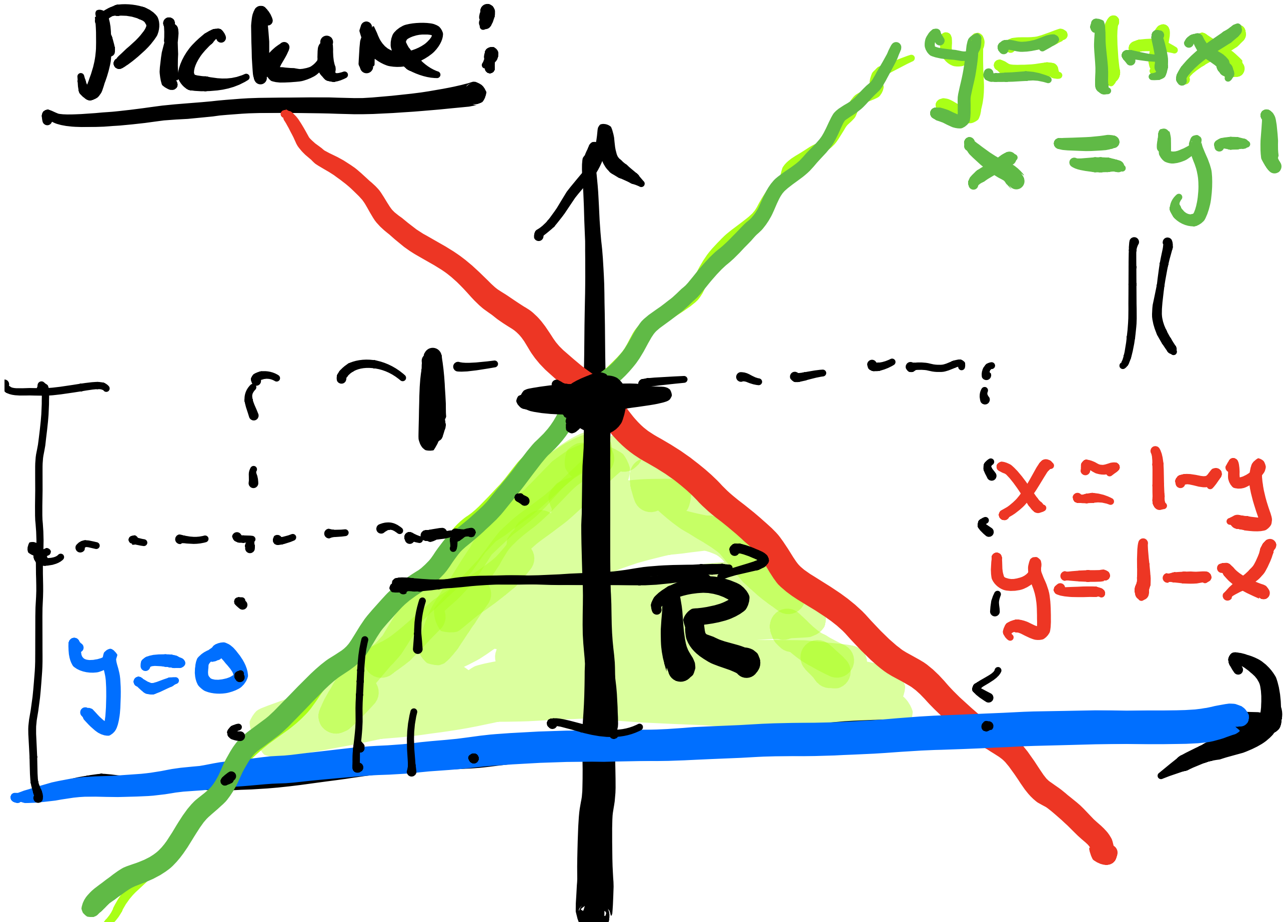
$\boxed{\text{Fubini!}}$?

No: rectangles

b) Problem

Find the integral of $f(x,y) = y^2x^2$ over the region R bound by $\boxed{y = 1+x}$ $\boxed{y = 1-x}$, $y = 0$

Picture:



Setting up the integral: left to right.

$$\int_0^1 \int_{y-1}^{1-y} x^2 y^2 \, dx \, dy$$

functions of y

Common trap here is to use the wrong function.

Result: $1/90$

not allowed: $\int_0^x \sin x \, dx$ → $\int_0^x \sin t \, dt = -\cos x$ ✓

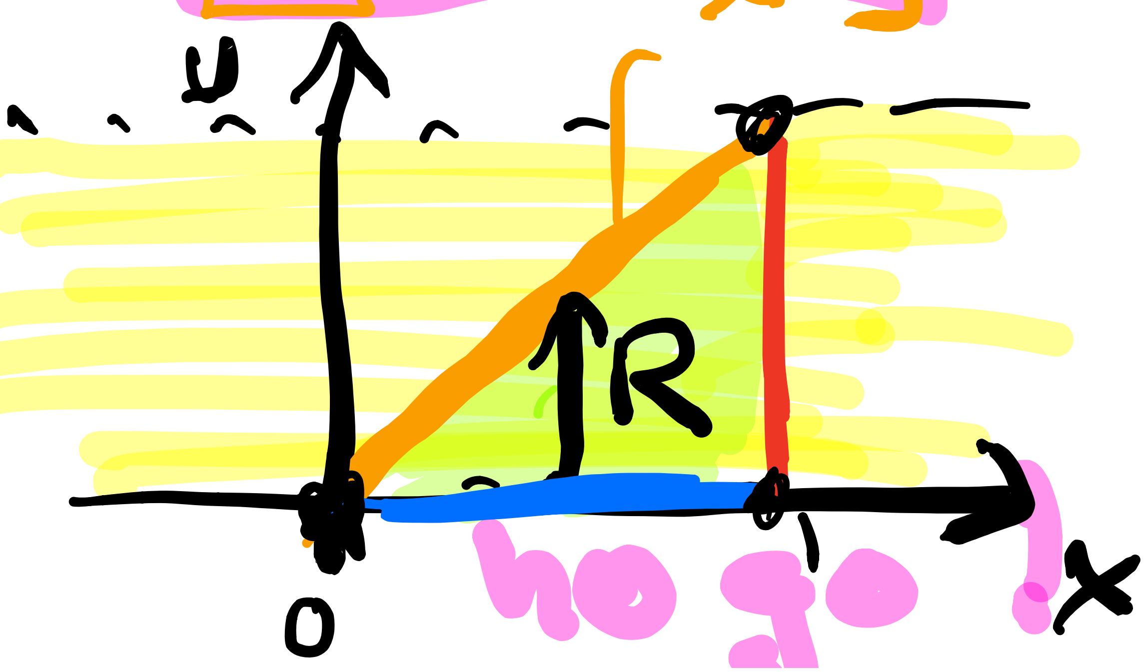
$$\int_{1-x} x^2 y^2 \, dx$$

abuse of notation

(5) Change order of integration

$$\int_0^1 \int_y^1 e^{-x^2}\, dx\, dy$$

$x = y$



Change the order of integral

$$\int_0^1 \int_0^x e^{-x^2}\, dy\, dx$$

$$y e^{-x^2} \Big|_0^x$$

$$\int_0^1 x e^{-x^2}\, dx$$

Substitution

$$u = x^2$$

$$du = 2x \, dx$$

$$dx = \frac{du}{2x}$$

$$\int_0^1 \frac{1}{2} e^{-u} \, du$$

$$-\frac{e^{-u}}{2} \Big|_0^1$$

$$= \frac{1}{2} - \frac{e^{-1}}{2}$$

$$= \boxed{\frac{1 - 1/e}{2}}$$

5min

Intermission