Unit 16

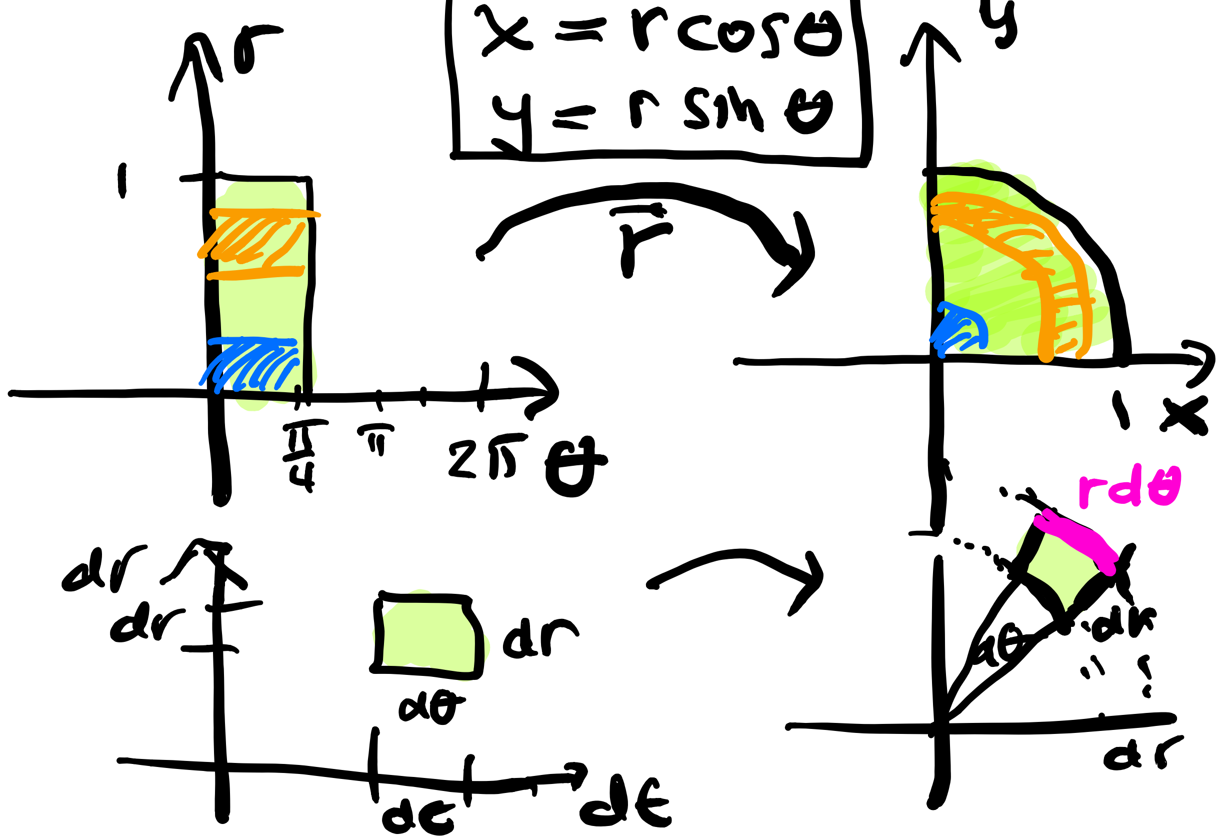
Polar and Surface integrals

1. Polar integration
2. Examples, Gifted
3. Surface integral
4. Examples

a) sphere
b) graphs
c) surface of revolution

1) Polar integral $x^2+y^2=r^2$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$



$dA = d\theta dr \qquad dA = \boxed{r}\, d\theta dr$

when going into polar coord
just add a factor $\boxed{r}$
" distortion factor "

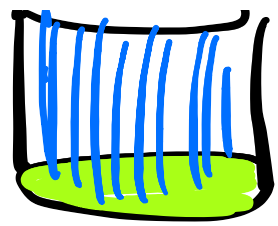
2 Examples

a) compute the area of the disk of radius L

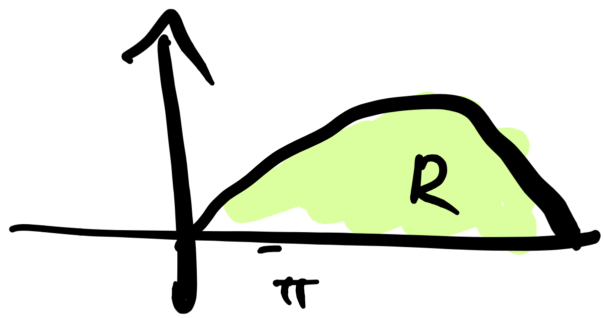
$$\int_0^{2\pi}\int_0^{L} 1\, \boxed{r}\, dr\, d\theta = \frac{2\pi L^2}{2} = \boxed{\pi L^2}$$

$\frac{L^2}{2}$

Area of R: $\displaystyle\iint_R 1 \, dA$!



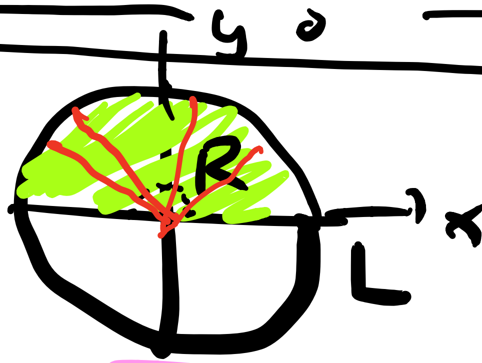
Volume = height · Area

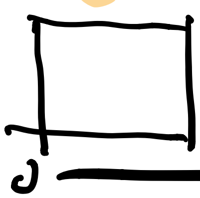


$$\int_0^{\pi}\int_0^{\sin x} 1\,dy\,dx \quad \text{Area}$$

$$= \int_0^{\pi} \sin x\,dx = 2$$

b)





$$\iint_R (x^2+y^2)\,dA$$

$$\int_0^{\pi}\int_0^{L} r^2 \,\boxed{r}\, dr\,d\theta$$

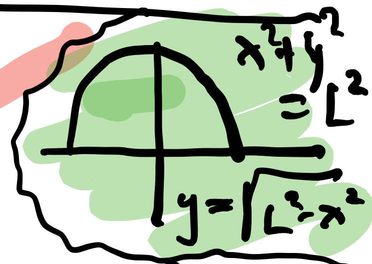
$$\frac{r^4}{4}\Big|_0^L = \frac{L^4}{4}$$

const for int $d\theta$

$$= \boxed{\pi L^4/4}$$

~~cartesian comput~~

$$\cancel{\int_{-L}^{L} \int_{0}^{\sqrt{L^2-x^2}} x^2+y^2 \, dy\, dx}$$



trig subst. problem

$$\int_a^b 1 \, dx = b-a$$

length

$$\iint_R 1 \, dA$$

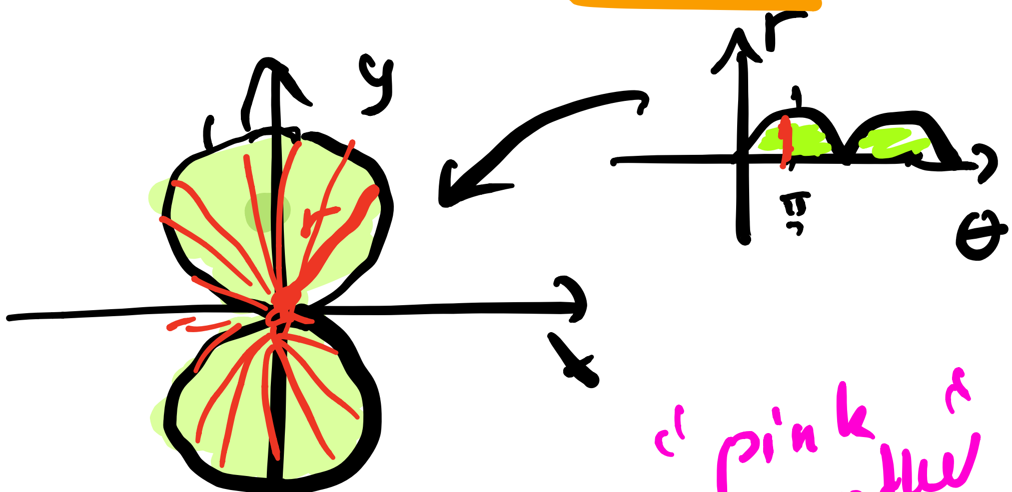
area

$$\iiint_R 1 \, dV = \text{Volume}$$

$\longrightarrow$ next week

c) $r \le \sin^2\theta$

Find the area.



$$\int_0^{2\pi}\int_0^{\sin^2\theta} 1\cdot \boxed{r}\, dr\, d\theta$$

"pink panther"

area

$\frac{r^2}{2}$

$$\int_0^{2\pi} \frac{\sin^4(\theta)}{2}\, d\theta$$

How to do this?

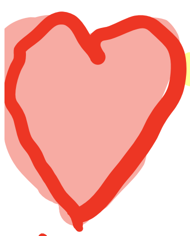
Trig identities:

The only thing you ever need to know:

Double angle formule

$$\cos^2(x) = \frac{1+\cos 2x}{2}$$

$$\sin^2(x) = \frac{1-\cos 2x}{2}$$



by heart

Add them: $\cos^2 x + \sin^2 x = 1$

Subtract them $\cos^2 x - \sin^2 x = \cos 2x$

differentiate the second $2\cos x \sin x = \sin 2x$

$$\int \cos^3 x \, dx$$

$$= \int \cos^2 x \cos x \, dx$$

$$= \int (1 - \sin^2 x) \cos x \, dx$$

$$= \int \cos x - \sin^2 x \cos x$$

$$\int \cos^4 x \, dx$$

$$= \int \cos^2 x \cos^2 x \, dx$$

$$\int \left(\frac{1 + \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)$$

double angle again!

d) Our favorite problem

$$? = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy$$

$$= \int_0^{2\pi}\boxed{\int_0^{\infty} e^{-r^2}\, r\, dr}\, d\theta$$

$$\left.\frac{e^{-r^2}}{2}\right|_0^{\infty} = \frac{1}{2}$$

$$= \boxed{\pi}$$

Punch line: This is

equal to $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\, dx\, dy$

$$\int_{-\infty}^{\infty} e^{-y^2} \boxed{\int_{-\infty}^{\infty} e^{-x^2}\, dx}\, dy$$

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2}\, dx} \cdot \boxed{\int_{-\infty}^{\infty} e^{-y^2}\, dy}$$

They are the same: so:

$$\Longrightarrow \boxed{\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}}$$

PDF

$$\frac{e^{-x^2}}{\sqrt{\pi}}$$

Gaussian distribution

Normal

Prob. density function

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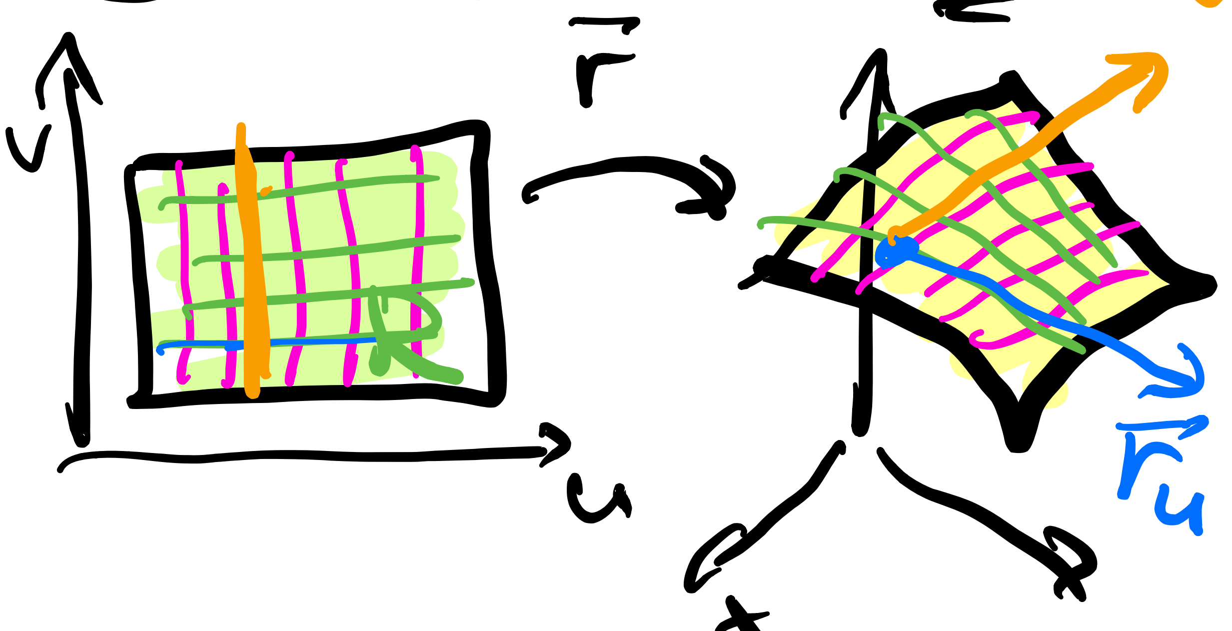
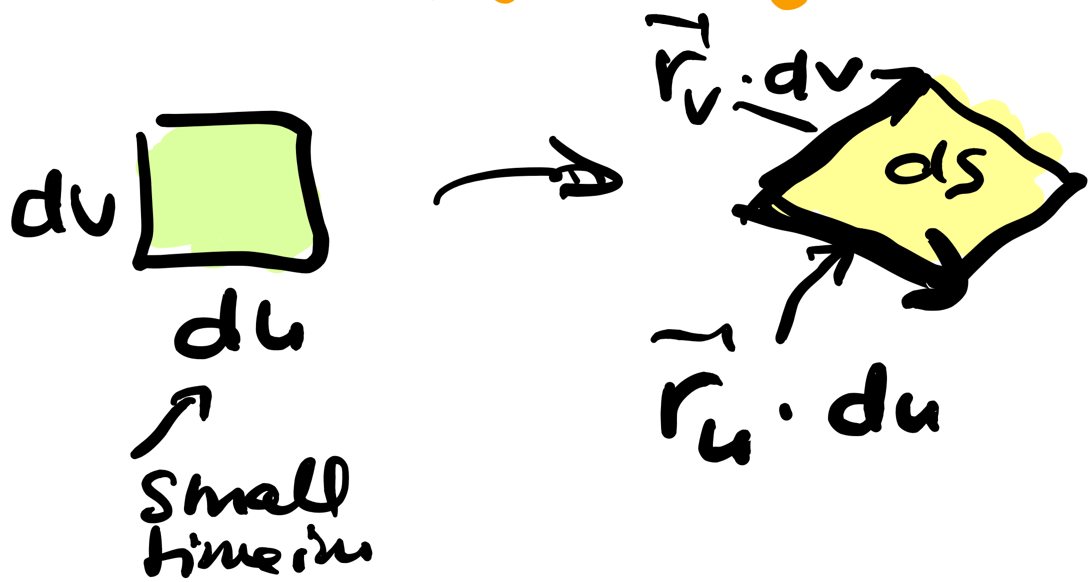
3 Surface area



$$\bar{r}(u,v)=\begin{bmatrix}x(u,v)\\ y(u,v)\\ z(u,v)\end{bmatrix}$$

$r_u = \begin{bmatrix} x_u \\ y_u \\ z_u \end{bmatrix}$ velocity vector of a grid curve

$\vec{r}_v = \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix}$ velocity of a grid curve



What is the area of ds ?

$$ds = |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

Surface area:

$$\iint_R |\vec{r}_u \times \vec{r}_v| \, du\,dv$$

double integral

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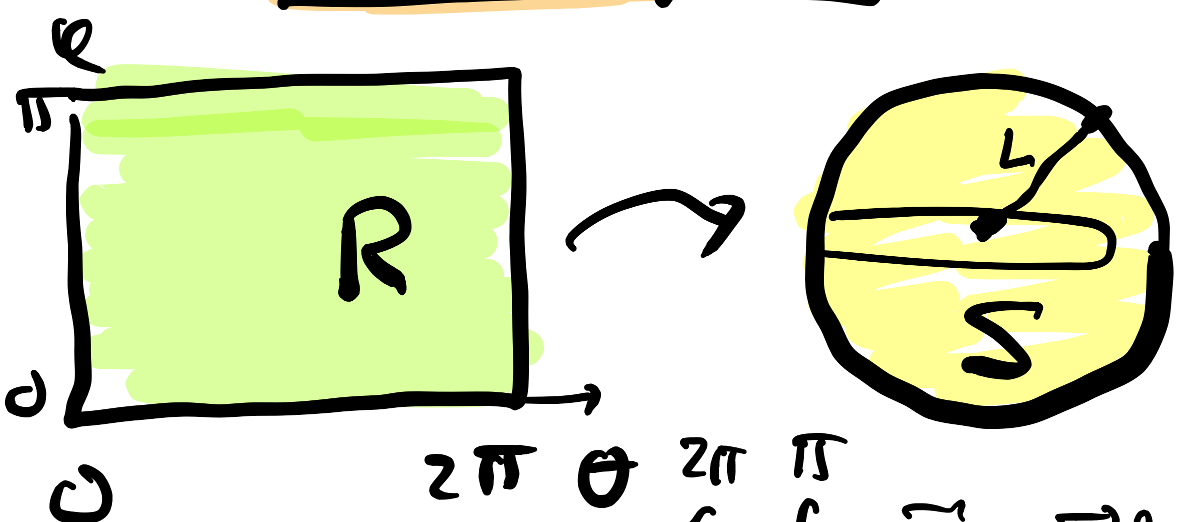
4) Examples



$$\int_0^{2\pi} \int_0^{\pi} |\vec{r}_\theta \times \vec{r}_\varphi| \, d\varphi \, d\theta$$

$$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} L\sin\varphi\cos\theta \\ L\sin\varphi\sin\theta \\ L\cos\varphi \end{bmatrix}$$

$$\vec{r}_\theta = \begin{bmatrix} -L\sin\varphi\sin\theta \\ L\sin\varphi\cos\theta \\ 0 \end{bmatrix}$$

$$\vec{r}_\varphi = \begin{bmatrix} L\cos\varphi\cos\theta \\ L\cos\varphi\sin\theta \\ -L\sin\varphi \end{bmatrix}$$

$$\begin{bmatrix} -L\sin\varphi\sin\theta \\ L\sin\varphi\cos\theta \\ 0 \end{bmatrix} \times \begin{bmatrix} L\cos\varphi\cos\theta \\ L\cos\varphi\sin\theta \\ -L\sin\varphi \end{bmatrix}$$

$$\begin{matrix} -L\sin\varphi\sin\theta & & L\cos\varphi\cos\theta \\ L\sin\varphi\cos\theta & & L\cos\varphi\sin\theta \end{matrix}$$

$$\begin{bmatrix} -L^2\sin^2\varphi\cos\theta \\ -L^2\sin^2\varphi\sin\theta \\ -L^2\cos\varphi\sin\varphi \end{bmatrix} \leftarrow \cos^2\theta + \sin^2\theta = 1$$

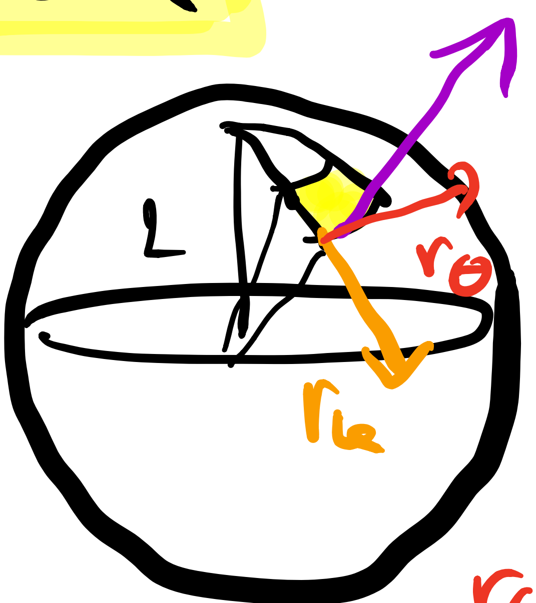
$$-L^2\sin\varphi \begin{bmatrix}\sin\varphi\cos\theta\\ \sin\varphi\sin\theta\\ \cos\varphi\end{bmatrix}$$

1

point on a unit sphere

The length is :

$$\boxed{L^2\sin\varphi}$$



$r_\varphi \times \vec{r_\theta}$ is perpendicular to the surface

r_θ and r_φ are tangent to the surface

$$\int_0^{2\pi}\int_0^{\pi} L^2\sin\varphi \quad d\varphi\, d\theta$$

surface area

$$\int_0^{2\pi} L^2 \underbrace{\boxed{\int_0^{\pi} \sin\varphi \, d\varphi}}_{2} \, d\theta$$

$$2L^2 \int_0^{2\pi} 1 \, d\theta = \boxed{4\pi L^2}$$

Archimedes.

φ, θ coordinates

Euler angles

Later

How did Archimedes do it?

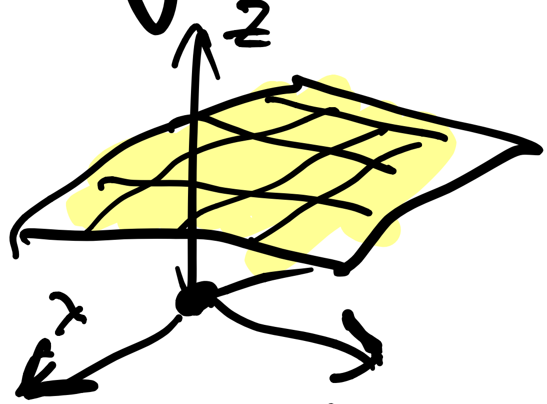
Compare with cylind

Cylinder area!

$$\underbrace{2\pi L}_{\text{circumference}} \quad \underbrace{2L}_{\text{height}}$$

$$\boxed{4\pi L^2}$$

b) $\begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}$ graph



$$|r_x \times r_y|$$

$$= \left| \begin{bmatrix} 1 \\ 0 \\ f_x \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ f_y \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix} \right|$$

$$\iint \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$$

R V

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c) Surface of revolution

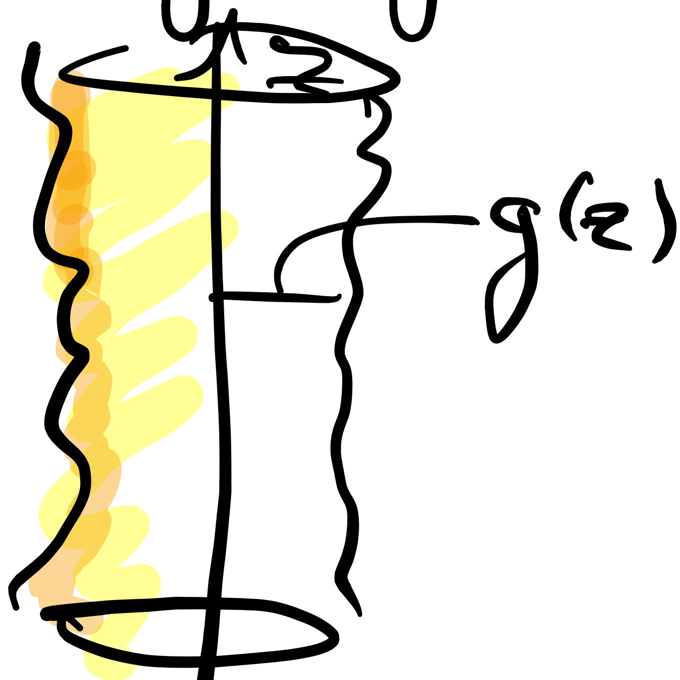
$$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} g(z)\cos\theta \\ g(z)\sin\theta \\ z \end{bmatrix}$$

$$\left| \vec{r}_\theta \times \vec{r}_z \right|$$

$$= \sqrt{1 + g'(z)^2}\; g(z)$$

2π

$$\int_0^{2\pi}\int_a^b \sqrt{1+g'(z)^2}\, g(z)\, dz\, d\theta$$



Example:

$$g(z) = \frac{1}{z}$$

gabriels trumpet

$g(z) = \frac{1}{z}$

radius

≥ 1

$$2\pi \int_{1}^{L} \sqrt{1+\left(\frac{1}{z^2}\right)^2}\,\frac{1}{z}\,dz$$

L

$$\geq 2\pi \int_1^{L} \frac{1}{z}\, dz$$

$$= \log(L) - \log(1)$$

$$= \log(L)$$

$$\downarrow$$

$$\infty$$

Fill with paint but can not paind the trumpet

volume $= \pi$