Unit 20 Line integrals

① Line integrals

Seagul

$\vec{F} = [y, 0]$

not a gradient field

$\vec{r}(t) = [\underbrace{\cos t}_{x}, \underbrace{\sin t + 3}_{y}]$

Curve. t = time

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$a \int$ Force · velocity dt

$= \int \text{Power} = \text{Energy}$

Example : $0 \le t \le 2\pi$

one loop.

$$\int_0^{2\pi} \left[\overset{-y}{-\sin t - 3}, \overset{0}{0}\right] \cdot \left[\overset{x'}{-\sin t}, \overset{y'}{\cos t}\right] dt$$

$$= \int_0^{2\pi} \sin^2 t + 3\sin t + 0 \, dt$$

$$= \int_0^{2\pi} \left(\frac{1 - \cos 2t}{2}\right) + 3\sin t \, dt$$

$$= \boxed{\pi}$$

Bird has gained energy

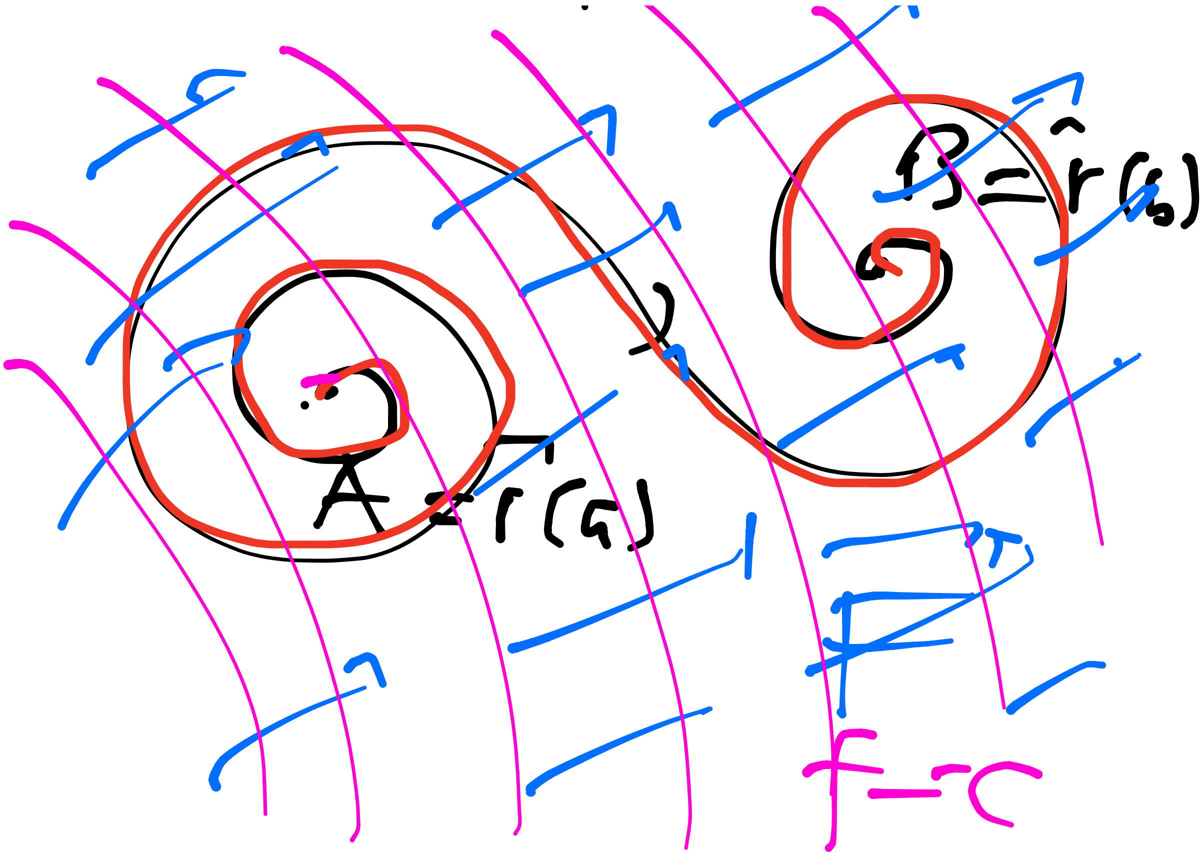
Fund thm Line Int

② FTLI

Theorem: If $\vec{F} = \nabla f$, then

$$\int_a^b \vec{F}(\vec{r}(t))\,\vec{r}'(t)\,dt = f(r(b)) - f(r(a))$$

f is the potential



$f(B) - f(A)$

is the work done when going from A to B

Proof:

$$\int_a^b \underbrace{\overset{\text{Chain rule}}{\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)}}_{\frac{d}{dt} f(\vec{r}(t))} \, dt$$

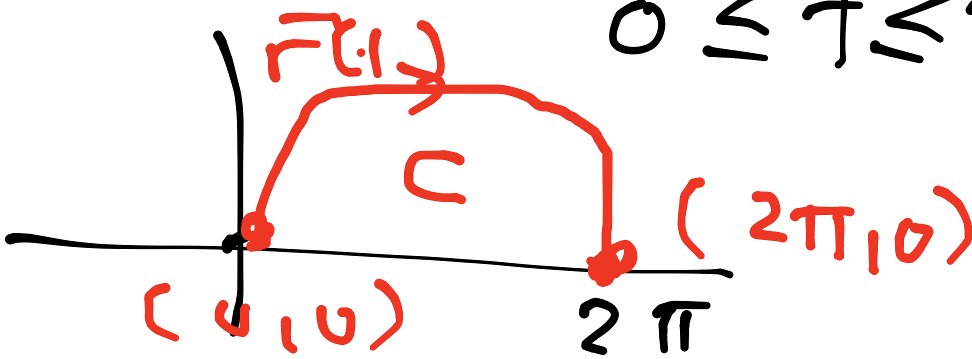
$$= \int_a^b \frac{d}{dt} f(\vec{r}(t)) \, dt$$

$$\overset{\text{FTC}}{=} f(\vec{r}(b)) - f(\vec{r}(a))$$

③ Example

$$\vec{r}(t) = [t, \sin''''(t)]$$

$$0 \le t \le 2\pi$$



$$\vec{F} = [\underbrace{x^{100} + y}_{f_x}, \underbrace{y^7 + x + 5}_{f_y}]$$

Find $\displaystyle\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$

$= \displaystyle\int \vec{F} \cdot d\vec{r}$ show

C hand

Solution: Find the potential f

$$f = \frac{x^{101}}{101} + xy + \frac{y^8}{8} + 5y$$

$$f(2\pi, 0) - f(0, 0)$$

$$\frac{(2\pi)^{101}}{101} - 0 = \boxed{\frac{2\pi^{101}}{101}}$$

The integral would be insane.

④ Conservative

closed curve A

grad field
$F = \nabla f$

closed loop pr.

B

A

Path ind.

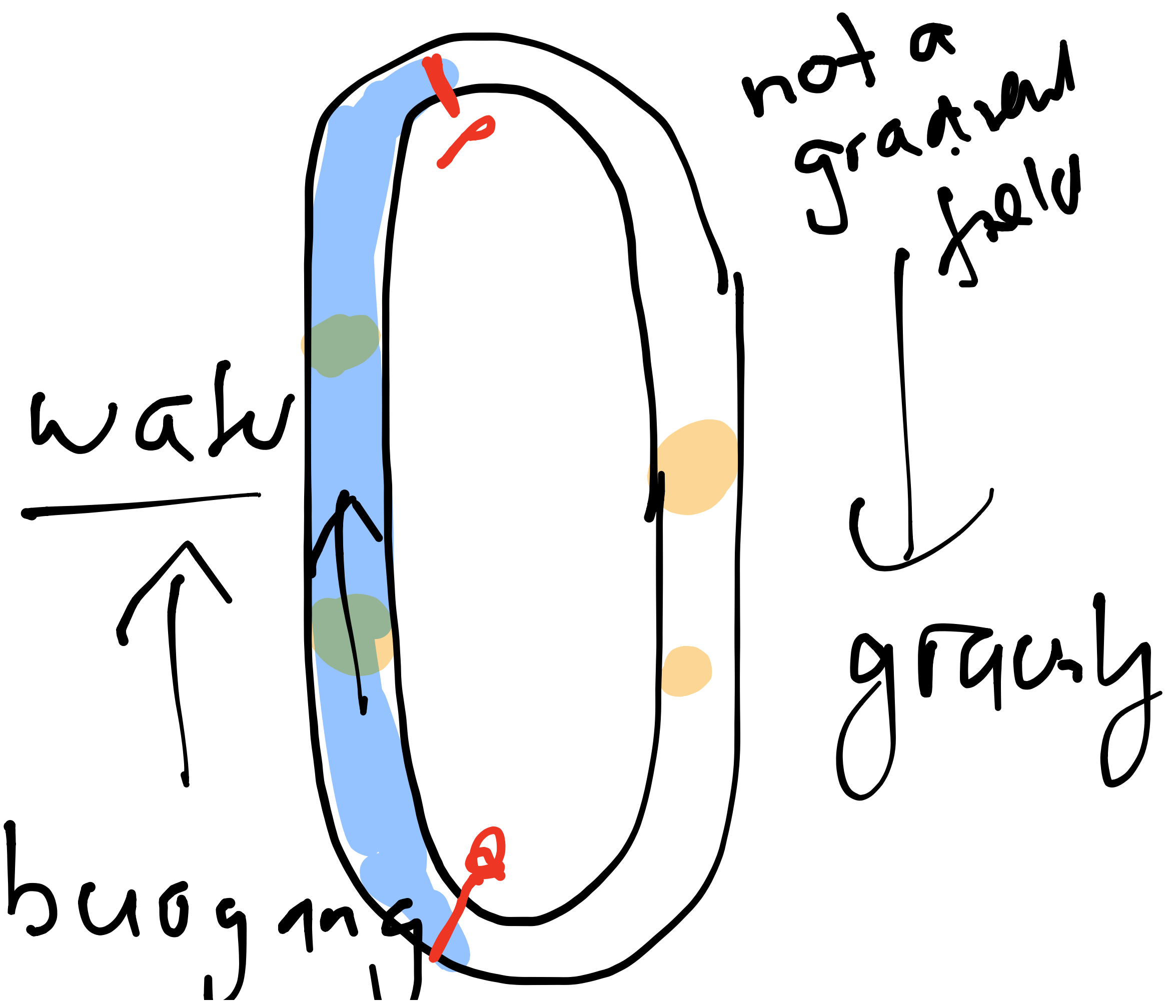
Conservative

$Q_x - P_y = 0$

if F is defined everywhere

Perpetual motion:

Knill lab



Perpetual motion