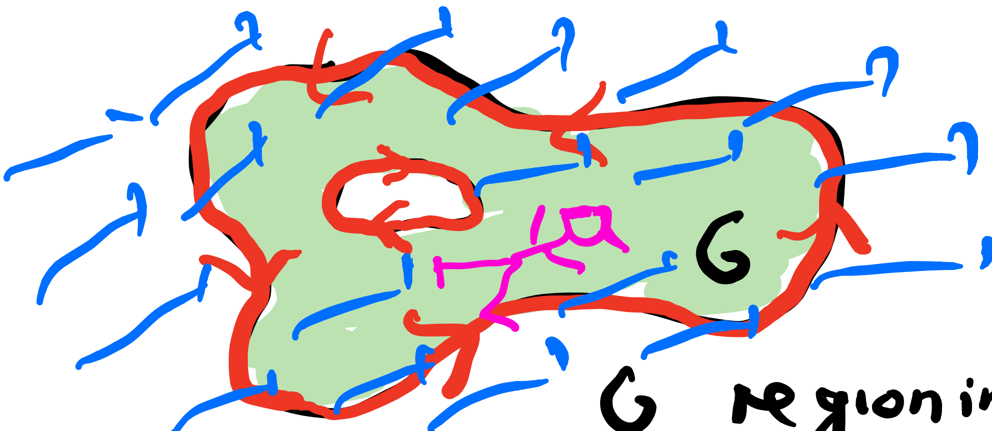
Unit 21

Green's theorem

1 Theorem

$\vec{F}$ vector field.



left foot is on G
right foot on boundary

G region in $\mathbb{R}^2$

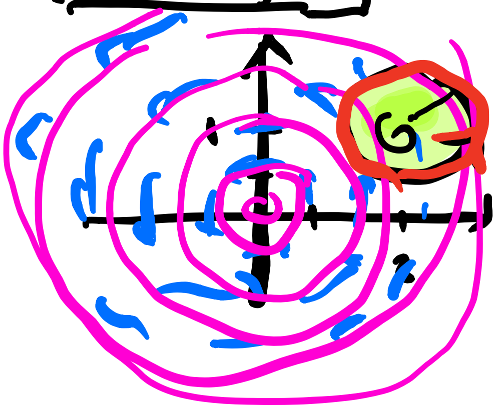
C boundary oriented, so that G is on the left

Green theorem

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \iint_G \operatorname{curl}(F)\, dA}$$

George Green : "Good will hunting"

Example:



$\vec{F} = [-y, x] = [P, Q]$

$G = \{ (x-2)^2 + (y-2)^2 \leq 1 \}$

circ C

radius

• **Left hand side:**

$$\vec{r}(t) = \begin{bmatrix} 2+\cos t \\ 2+\sin t \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$0 \leq t \leq 2\pi$$

$$\int_0^{2\pi} \begin{bmatrix} -2-\sin t \\ 2+\cos t \end{bmatrix} \cdot \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} dt =$$

$$= \int_0^{2\pi} 2\sin t + \underbrace{\sin^2 t + 2\cos t + \cos^2 t}_{1}\, dt$$

$$= \int_0^{2\pi} 2\sin t + 2\cos t \, dt + \int_0^{2\pi} 1 \, dt$$

$$= \boxed{2\pi}$$

• **Right hand side:** $\iint_G \mathrm{curl}(F)\, dA$

$$\mathrm{curl}(\vec{F}) = Q_x - P_y$$
$$= 1 - (-1) = 2$$

$$\iint_G \mathrm{curl}(\vec{F})\, dA = \int 2 \, dA$$

$$= 2 \int_G 1 \, dA = 2\,\text{Area}$$
$$= 2\pi$$

$$\left(\iint_G 1 \, dA = \text{Area} = \int_0^{2\pi} \int_0^1 r \, dr \, d\theta = \frac{1}{2} 2\pi = \pi \right)$$

---

② <u>Remarks</u>

i) If $\vec{F} = \nabla f = [f_x, f_y]$ then both sides are zero:

---

why? $\int_C \vec{F} \cdot d\vec{r} = 0$ because of <u>closed loop property</u>.

which followed from FTLI

$$\int_C \nabla f \, d\vec{r} = f(B) - f(A)$$

if $A = B$, then this is zero

$$\oint_C \nabla f \, d\vec{r} = 0$$

---

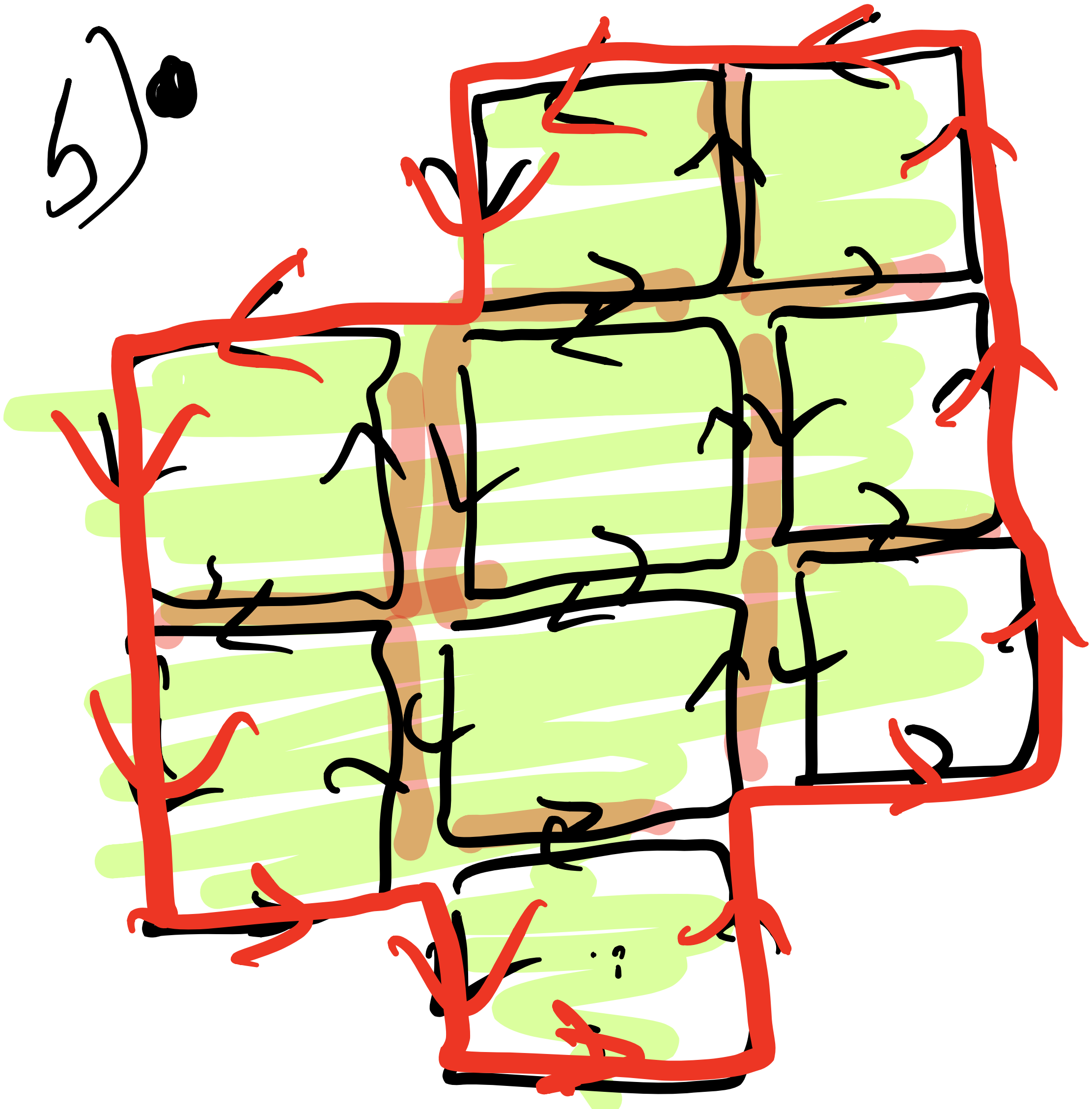
why is $\displaystyle\iint_G \mathrm{curl}(F)\, dA = 0$?

<u>Clairaut</u>:

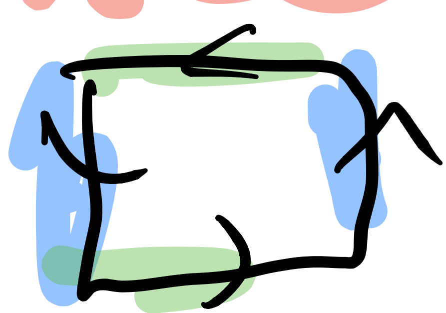
$$\mathrm{Curl}\left(\left[\overset{P}{f_x}, \overset{Q}{f_y}\right]\right)$$

$$Q_x - P_y = f_{yx} - f_{xy} = 0$$

5)



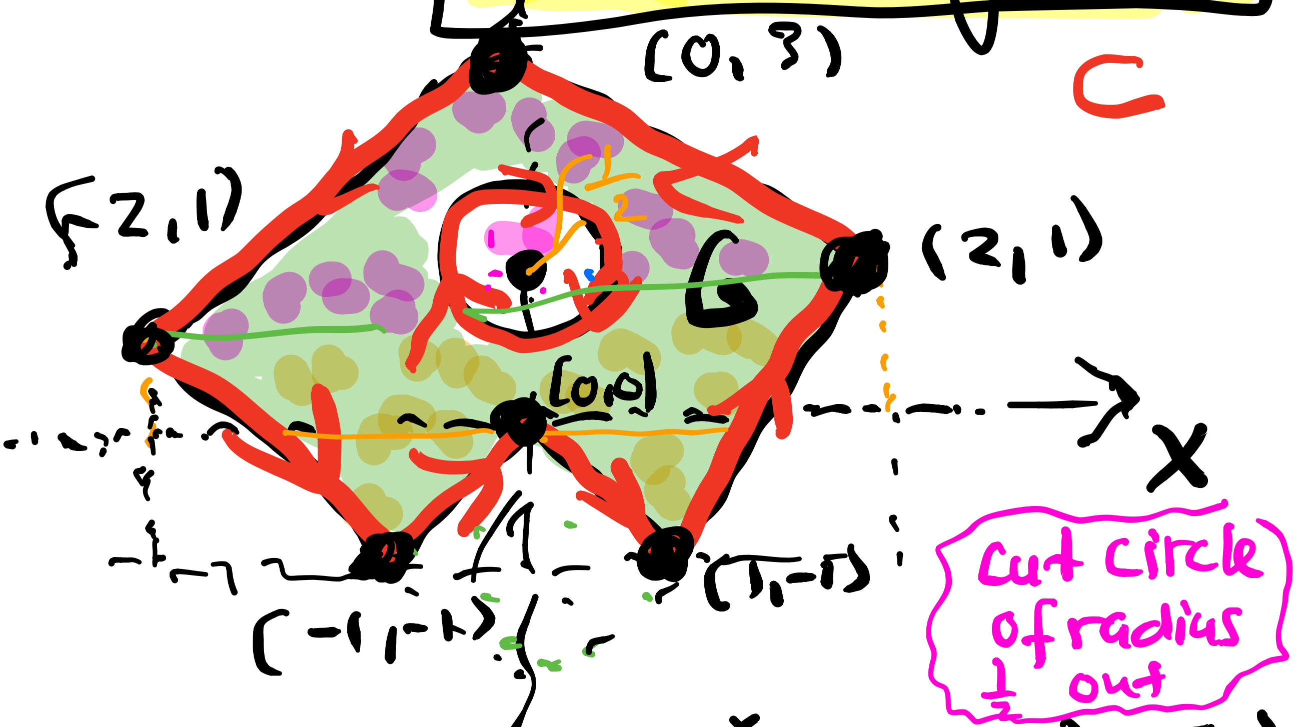
cancel



Q_x

P_y

3 How to compute line integrals



$$\vec{F} = \begin{bmatrix} \sin(e^x \sin(\sin x)) + 5y \\ 5e^{e^{e^{e^{e^y}}}} - 3x + 7y^7 \end{bmatrix}$$

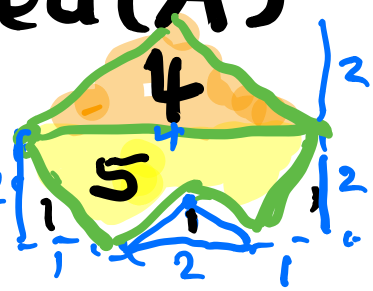
<u>Task</u>: Find $\int_C \vec{F} \cdot d\vec{r}$!

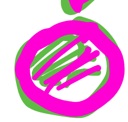
The answer is obtaind

using Green! $\quad curl(\vec{F}) = \boxed{-8}$

$$\iint_G curl(F)\, dA = \iint_G (-8)\, dA$$

(-8) Area(A)

$= (-8) \Big($ 

$-$  $\Big)$

$\frac{\pi}{4}$

$= (-8)\left(9 - \frac{\pi}{4}\right)$

$= \boxed{-72 + 2\pi}$

$8 - 3 = 5$

$+$

$\frac{2 \cdot 4}{2} = 4$

$= 9$

④ Area computation

$\vec{F}$ with $curl(\vec{F}) = 1$

allows to compute area:

$$\iint_G 1 \, dA = \iint_G \text{curl}(F) dA$$

$$= \int_C \vec{F} \cdot dr$$

<u>Theorem</u>

We can compute a line integral!

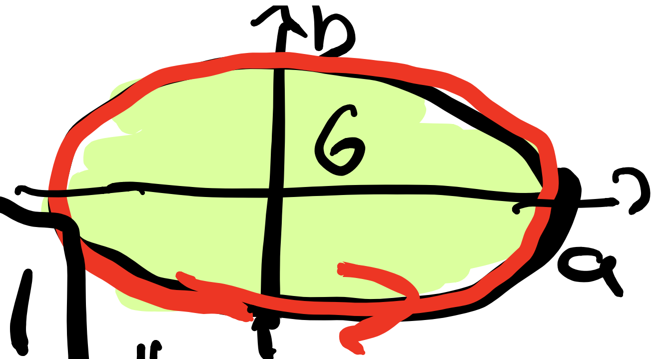
What vector field do we use?

$\boxed{\vec{F} = \begin{bmatrix} 0 \\ x \end{bmatrix}}$, $\vec{F} = \begin{bmatrix} -y \\ 0 \end{bmatrix}$

$\vec{F} = \begin{bmatrix} -y \\ x \end{bmatrix} / 2$ <u>examples</u>

Example:

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1}$$



"hardest geometry problem"
(Rushmore movie)

$$\vec{r}(t)=\begin{array}{cc}\begin{bmatrix} a\cos t \\ b\sin t\end{bmatrix} & \begin{array}{c} x \\ y\end{array}\end{array}$$

$$\vec{F} = [0, x] = \begin{bmatrix}0\\x\end{bmatrix}$$

$$\int_0^{2\pi} \begin{bmatrix}0\\a\cos t\end{bmatrix}\cdot\begin{bmatrix}-a\sin t\\ b\cos t\end{bmatrix} dt$$

$$=\int_0^{2\pi} ab\cos^2 t \, dt$$

$$=\int_0^{2\pi} ab\left(\frac{1+\cos 2t}{2}\right)dt = \boxed{ab\pi}$$

why F ??

$$\left[e^{e^x},\ e^{e^y}+x\right]$$

has also curl 1.

Laziness

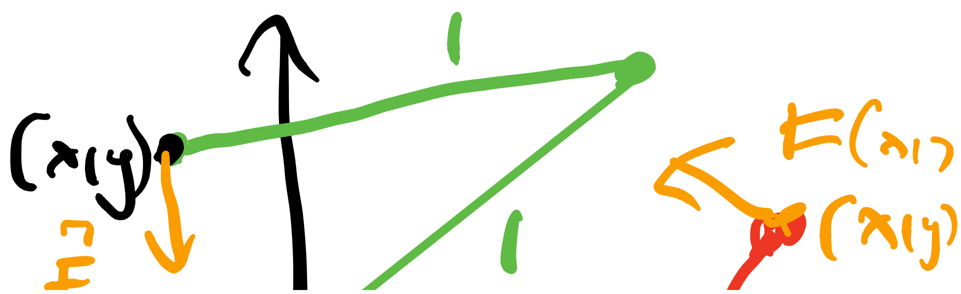
$$F=\begin{bmatrix}-y\\ x\end{bmatrix}\Big/2$$

also popular!

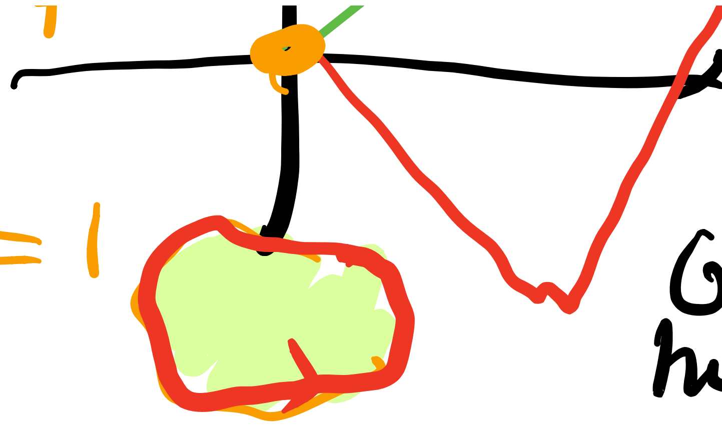
$$\int_0^{2\pi}\frac{1}{2}\begin{bmatrix}-b\sin t\\ a\cos t\end{bmatrix}\cdot\begin{bmatrix}-a\sin t\\ b\cos t\end{bmatrix}dt$$

$$\frac{ab}{2}\left(\cos^2 t+\sin^2 t\right) \Rightarrow \boxed{ab\pi}$$

Planimeter : mechanical computer



$ca(\overline{F}) = 1$



Green's theorem applicat.