Unit 22

curl, div, flux

this is a tool kit for Stokes and divergence theorem

$\begin{bmatrix}3\\4\end{bmatrix} \times \begin{bmatrix}1\\2\end{bmatrix} = \underbrace{6-4}_{=2}$

1 Curl in 2D

$\tilde{F} = \begin{bmatrix}P\\Q\end{bmatrix}$, $\text{curl}(F) = Q_x - P_y$

$\boxed{\nabla \times \vec{F}}$, $\nabla = \begin{bmatrix}\partial_x\\\partial_y\end{bmatrix}$

$\begin{bmatrix}\partial_x\\\partial_y\end{bmatrix} \times \begin{bmatrix}P\\Q\end{bmatrix} = Q_x - P_y$

Reminder:

$\boxed{\text{curl}(\text{grad}(f)) = 0}$

$\nabla \times \underline{\underline{\nabla}} f_0$

② $\boxed{\text{Curl in 3D}}$

$$\vec{F} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

$$\boxed{\operatorname{Curl}(\vec{F}) = \begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix}}$$

$$\nabla \times \vec{F} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

$$= \begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix}$$

---

a) $$\begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} x^2 y \\ y^2 z + x \\ x + y \end{bmatrix}$$

$$= \begin{bmatrix} 1 - y^2 \\ 0 - 1 \\ 1 - x^2 \end{bmatrix} = \begin{bmatrix} 1 - y^2 \\ -1 \\ 1 - x^2 \end{bmatrix}$$

b) $\text{curl}\left(\begin{bmatrix} -y \\ x \\ 1 \end{bmatrix}\right)$

tornado $\vec{F}$

$$\text{curl}(\vec{F}) = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} -y \\ x \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$ constant curl

---

HW: $\boxed{\text{curl}(\text{grad} f) = \vec{0}}$

$$\begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} f_{zy} - f_{yz} \\ f_{xz} - f_{zx} \\ f_{yx} - f_{xy} \end{bmatrix}$$

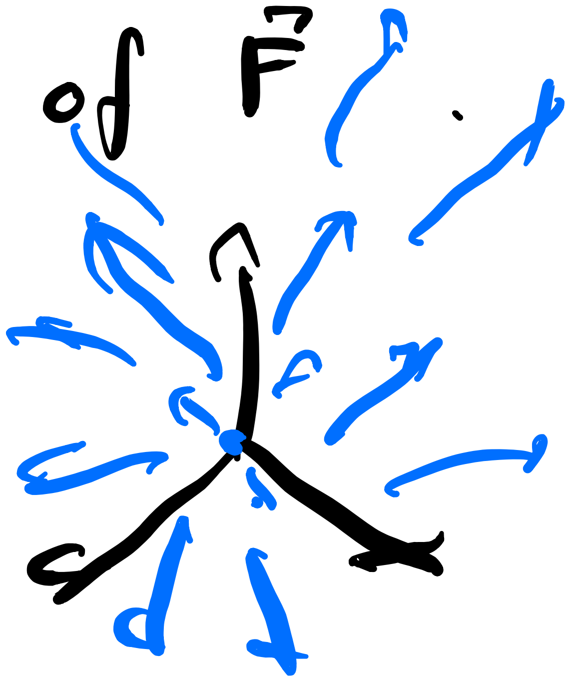
$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

---

3 $\boxed{\text{Divergence}}$

$$\text{div}\left(\begin{bmatrix}P\\Q\\R\end{bmatrix}\right) = P_x + Q_y + R_z$$

divergence measures expansion of $\vec{F}$



$$\vec{F} = \begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$\text{div}\,F = 3$$

$$\boxed{\text{div}\,\text{curl}(\vec{F}) = 0}$$

$$\text{div}\,\vec{F} = \begin{bmatrix}\partial_x\\\partial_y\\\partial_z\end{bmatrix}\cdot\begin{bmatrix}P\\Q\\R\end{bmatrix}$$ Dot

Proof:

$$\begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \cdot \begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix}$$

$$= \cancel{R_{yx}} - \cancel{Q_{zx}} + \cancel{P_{zy}} - \cancel{R_{xy}} + \cancel{Q_{xz}} - \cancel{P_{yz}} = 0$$

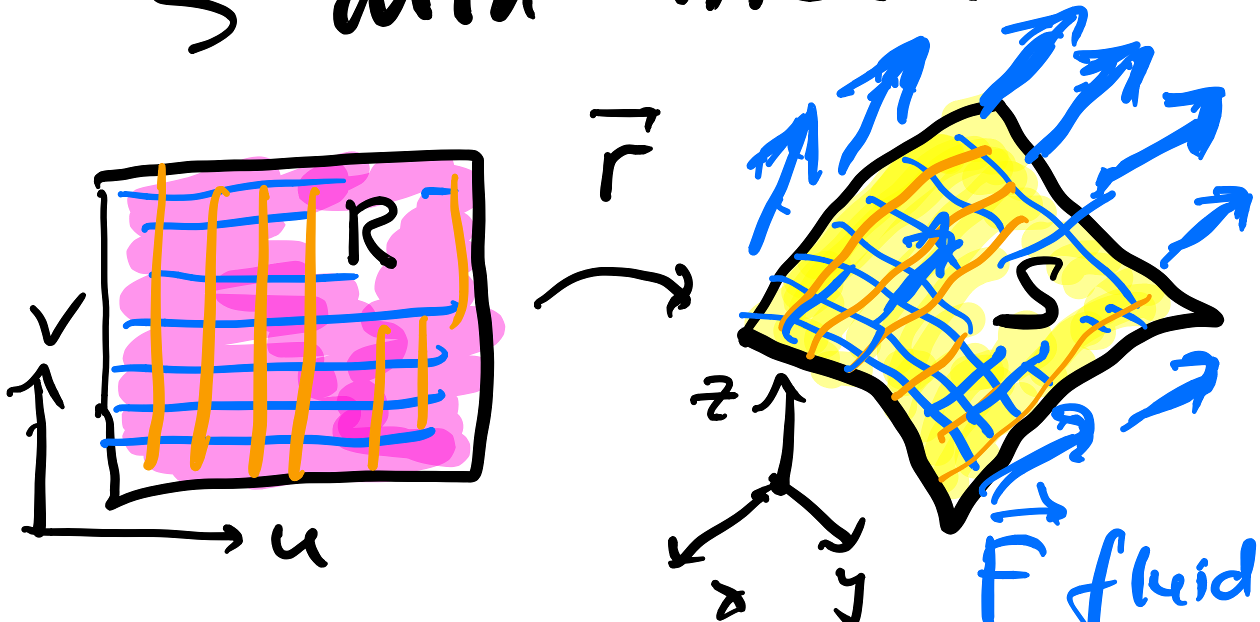
Derivatives in calculus

$$\begin{tikzcd} 1 \arrow[r, "\frac{d}{dx}"] & 1 & & \\ 1 \arrow[r, "grad"] & 2 \arrow[r, "curl"] & 1 & \\ 1 \arrow[r, "grad"] & 3 \arrow[r, "curl"] & 3 \arrow[r, "div"] & 1 \end{tikzcd}$$

Scalar fields vector field.

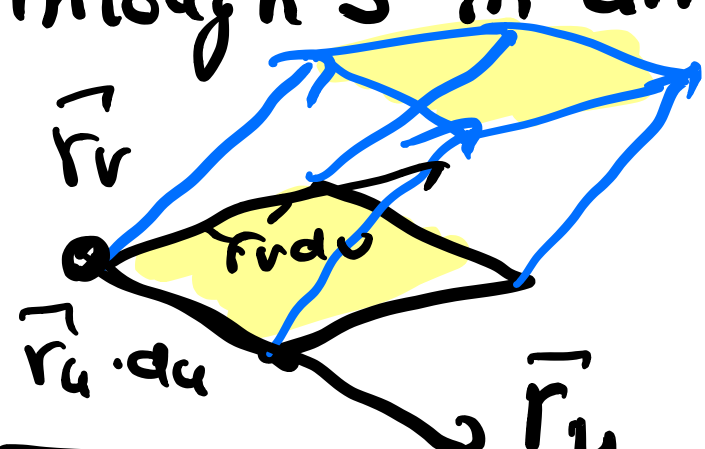
(4) Flux integral

5 min break.



$\vec{F}$ fluid field

How much field passes through S in unit time?



volume of a parallelepiped

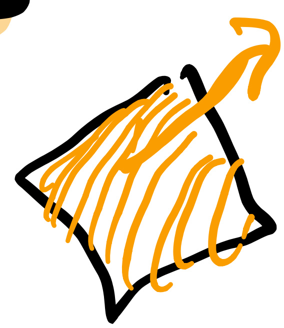
$\boxed{\vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \, du \, dv}$ is

Flux of $\vec{F}$ through S is defined as

$$\boxed{\iint_R \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \, du \, dv}$$

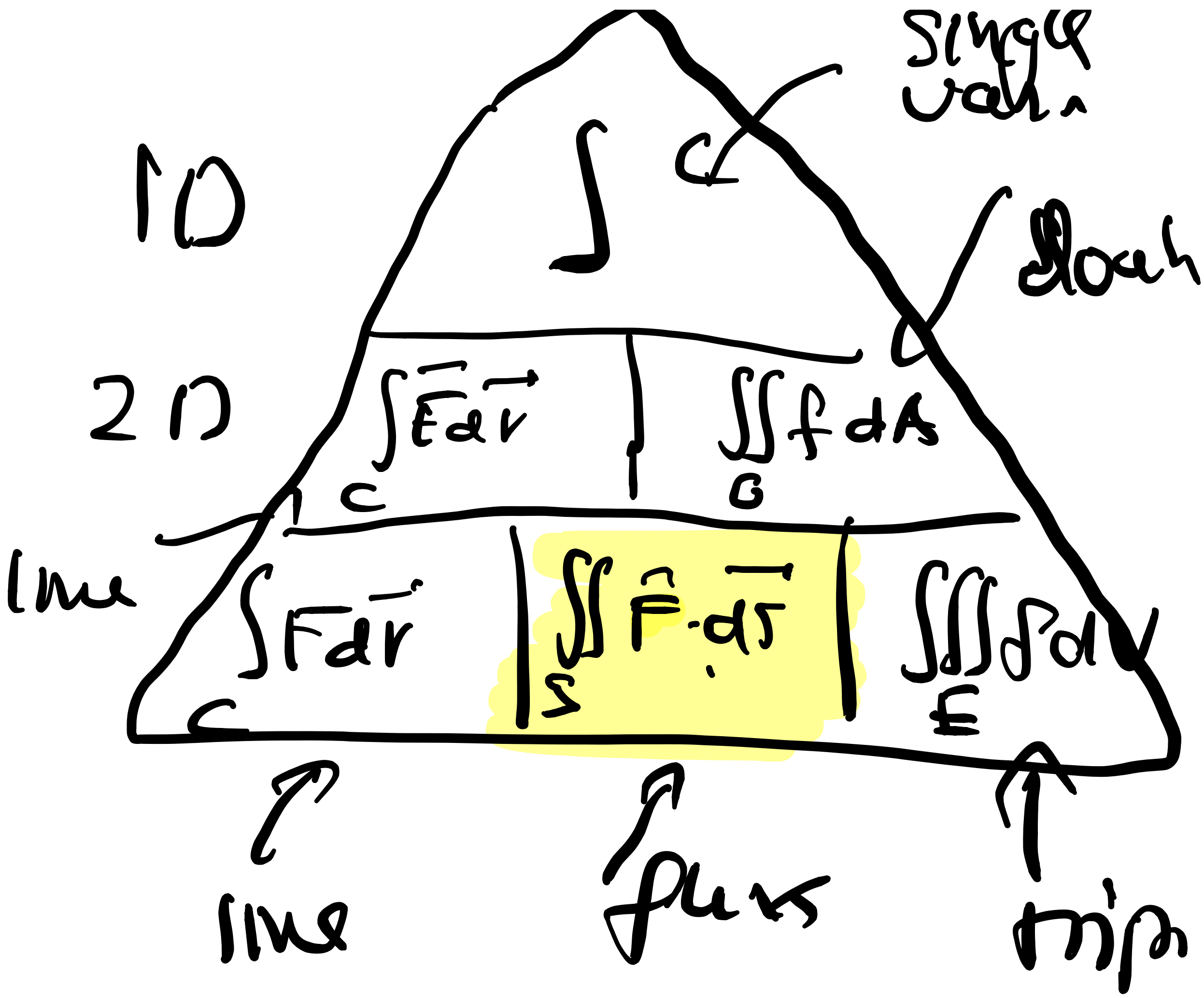
$$= \iint_S \vec{F} \cdot \vec{dS}$$

$\vec{dS}$



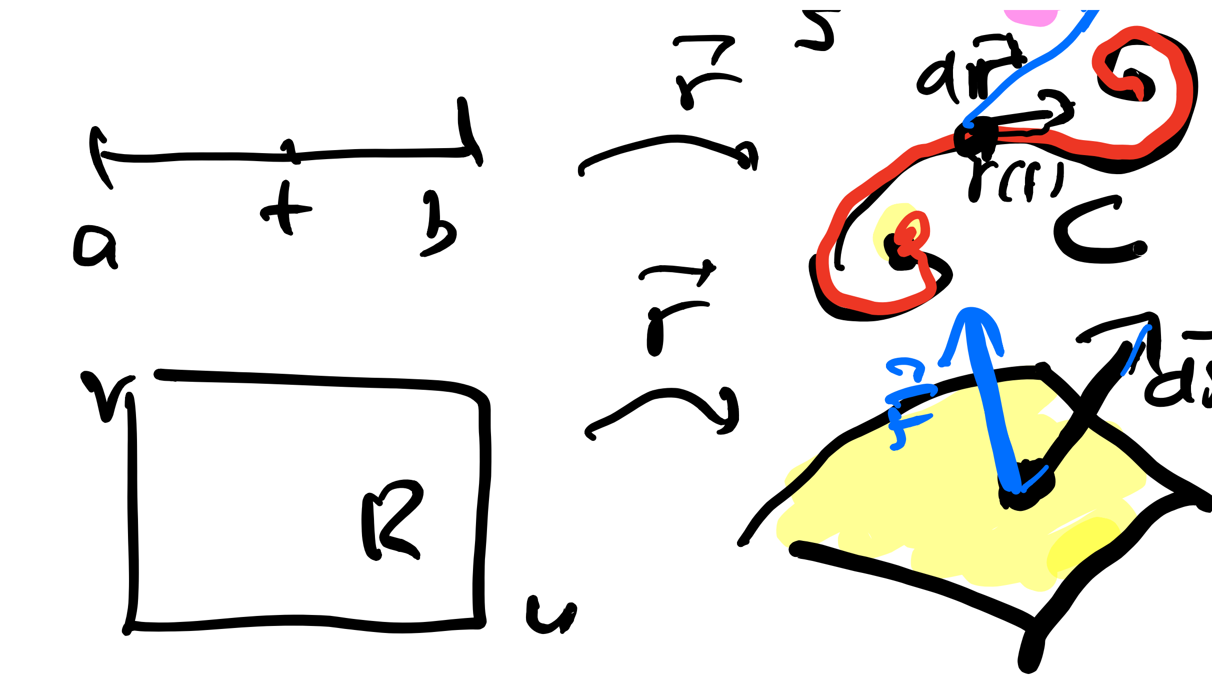
Compare with surface area

$$dS = |\vec{dS}|$$
$$= |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

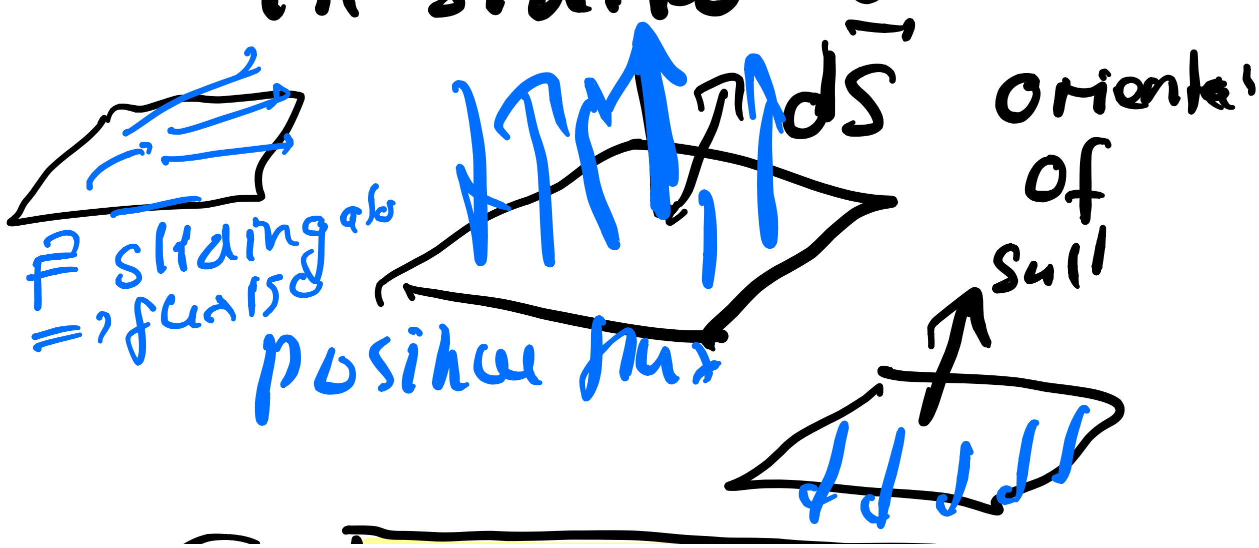


$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_C \vec{F} \cdot d\vec{r}$$

$$\iint_R F(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \,du\,dv = \iint_S \vec{F} \cdot d\vec{S}$$



The sign of the flux is positive if $\vec{F}$ and $d\vec{S}$ are pointing in similar direction



5 Examples

a)

S sphere oriented outward

$x^2+y^2+z^2 \leq 1$

$$\vec{F} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$$

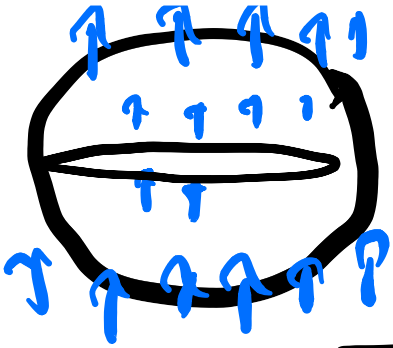
Check your intuition!

$$\vec{r}(\varphi,\theta) = \begin{bmatrix} \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \\ \cos\varphi \end{bmatrix}$$

defines the orientation

Is the flux of F through S ?

~~Positive~~	Negative	0



$$\vec{F} = \begin{bmatrix} 0 \\ 0 \\ z^2 \end{bmatrix}$$

always up

$\Longrightarrow$ zero

The angle between $\vec{r}_u \times \vec{r}_v$ and $\vec{F}(\vec{r}(u,v))$ is always acute or perpendicular (equator)

$$\int_0^{2\pi}\int_0^{\pi} \begin{bmatrix} 0 \\ 0 \\ \cos\varphi \end{bmatrix} \cdot \begin{bmatrix} \sin^2\varphi\cos\theta \\ \sin^2\varphi\sin\theta \\ \sin\varphi\cos\varphi \end{bmatrix} d\varphi\, d\theta$$

$$\vec{r}(\varphi,\theta) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \\ \cos\varphi \end{bmatrix}$$

$$\boxed{\vec{r}_\varphi \times \vec{r}_\theta = \sin\varphi \begin{bmatrix} \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \\ \cos\varphi \end{bmatrix}}$$

2π /

$$= \int_0^{2\pi} \boxed{\int_0^{\pi} \sin\varphi \cos^2\varphi \, d\varphi} \, d\theta$$

$$-\frac{\cos^3(\varphi)}{3}\Big|_0^{\pi} = \frac{2}{3}$$

$$= \int_0^{2\pi} \frac{2}{3} \, d\theta = \boxed{\frac{4\pi}{3}} = \text{Volume}$$

(look ahead : $\operatorname{div} F = \operatorname{div} \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = 1$)

$\operatorname{div}(\vec{F})(x,y,z)$ tells how much field is created at (x,y,z)

$\longrightarrow$ Divergence thm will make this clear.

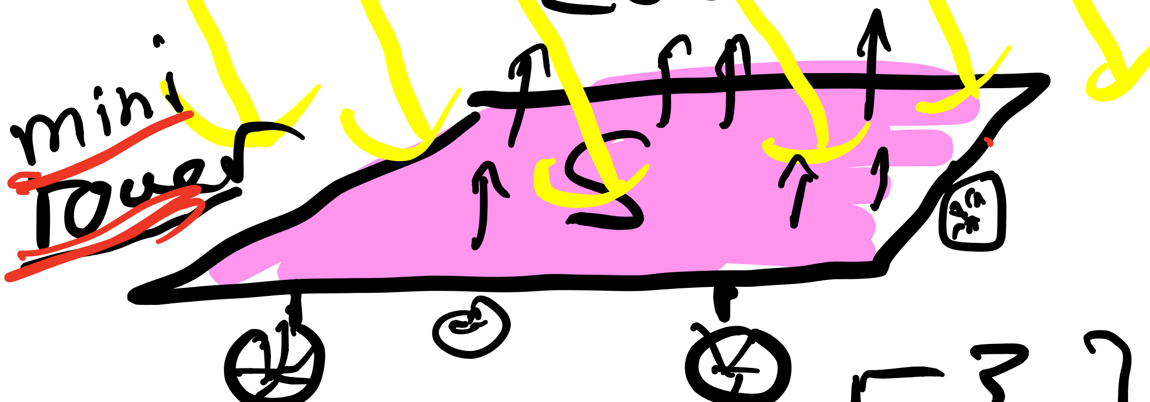
b) Moon Tower

My property on the moon size of manhattan for 80 Dollars.

knill estate !

"Lunarembassy"

$$\vec{r}(u,v) = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}, \quad |u| \le 1, \ |v| \le 1$$



$$\vec{F} = \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}$$

Flux of $\vec{F}$ through S^-

$$\int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dv\,du$$

$$\vec{r}_u \times \vec{r}_v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \int_{-1}^{1}\int_{-1}^{1} (-5)\, du\,dv = \boxed{-20}$$

The orientation of S is given by the parametrization.

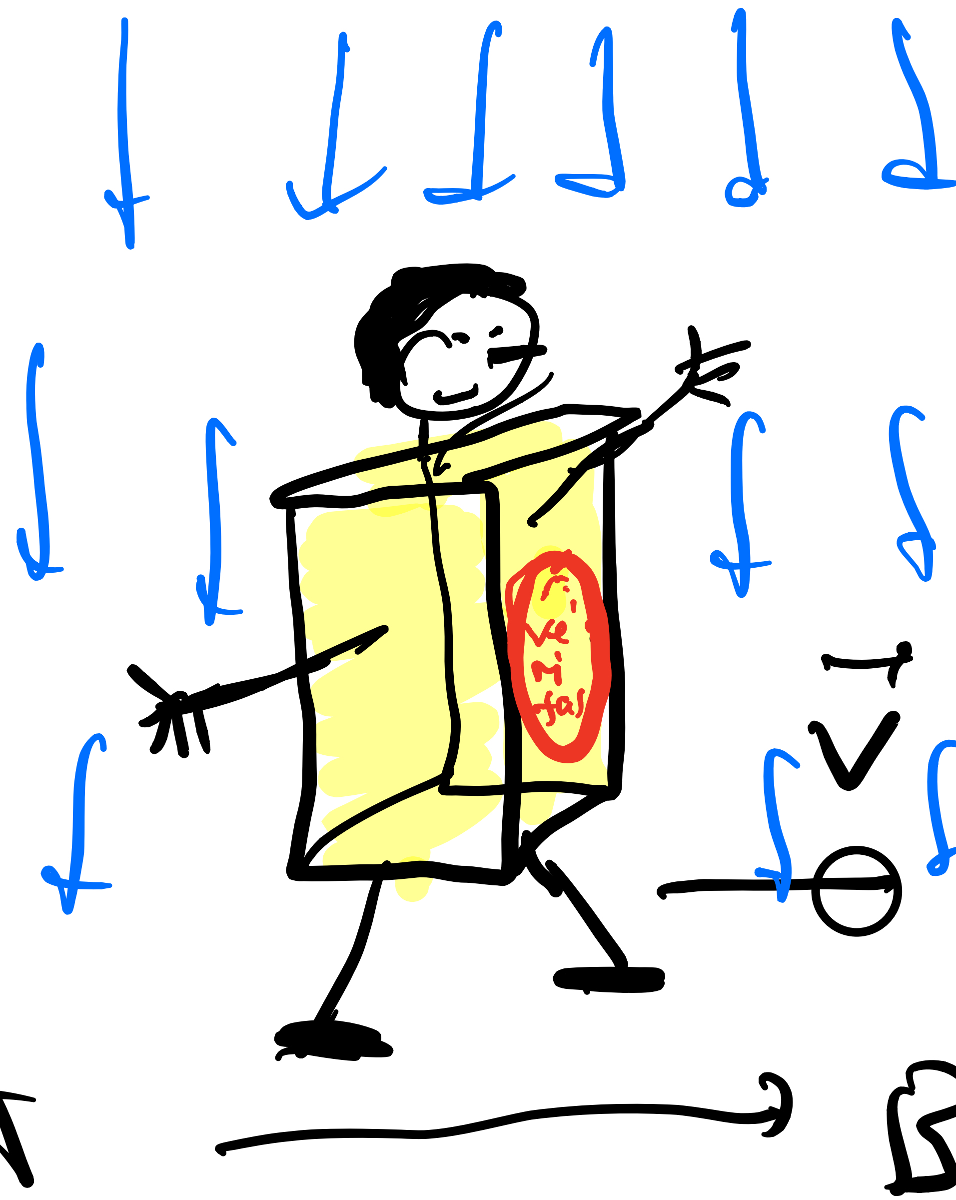
The orientation direction is given by $\vec{r_u} \times \vec{r_v}$

The problem usually gives you the orientation.

One has just to be aware of the given orientation when computing the flux.

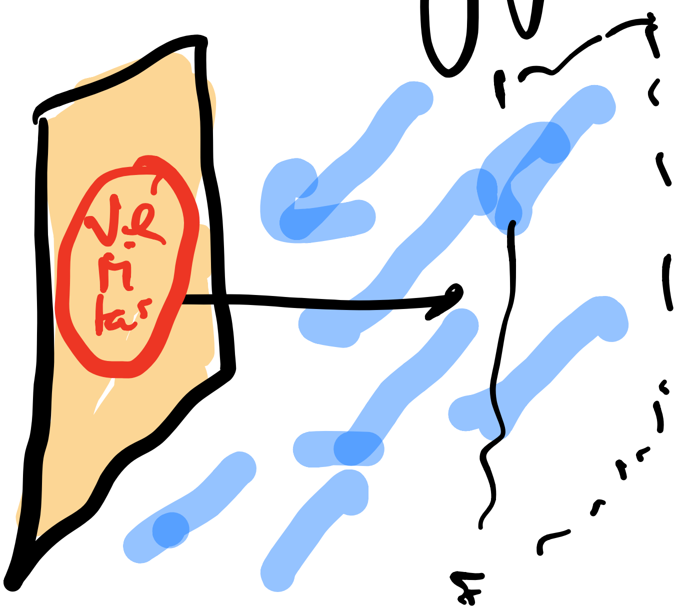
C)

Rain problem



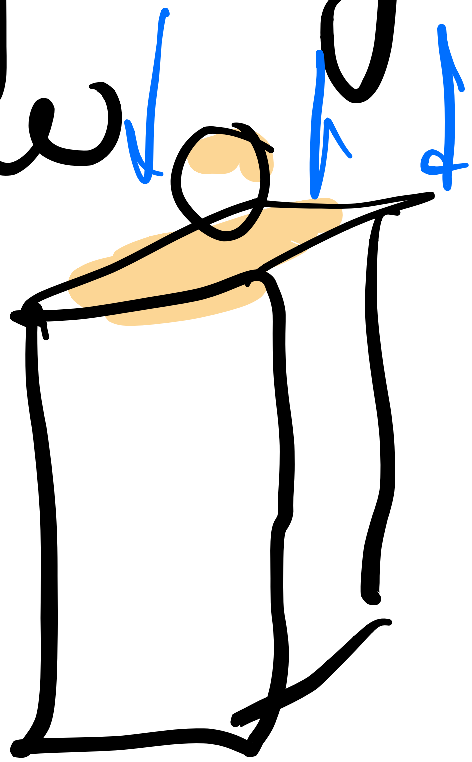
Is it better to go fast? you

get Soaked by rain coming from the front



If you go slow

then you only



get wet from the top. But you are also longer in the rain!

You compute the Flux of $\vec{F}$ through S and multipy by time.

Flux appears also in electromagnetism

Flux of magnetic field.

for example.