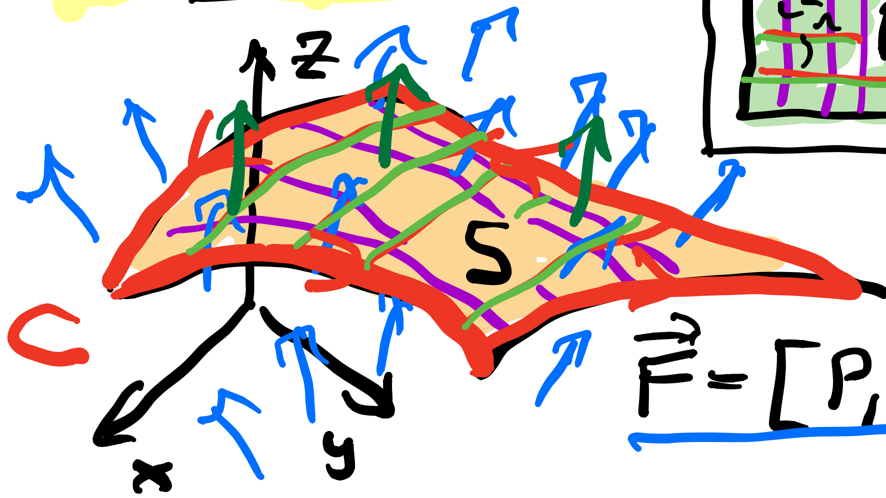
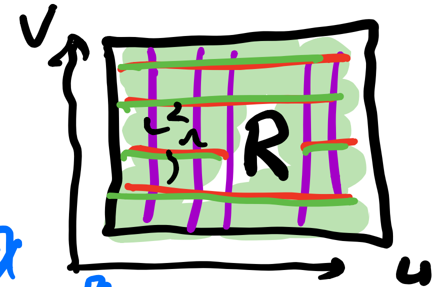
Unit 23

Stokes theorem

(1) Theorem





$\vec F = [P, Q, R]$

The surface S is oriented by the parametrization $\vec r_u \times \vec r_v$ which tells what part of S is "up". Also the boundary curve C is oriented. Like in Green, the surface is to our left

$$\boxed{\iint_S \mathrm{curl}(\vec F)\cdot d\vec S = \int_C \vec F \cdot d\vec r}$$

Stokes

In detail: written out.

$$\boxed{\iint_R \operatorname{curl}(\vec{F})(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \, du\, dv = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt}$$

Lots of things come together

For us, the primary goal, is to use this theorem to compute things.

<u>One remark!</u> What happens if $F = \nabla f$ is a gradient field.

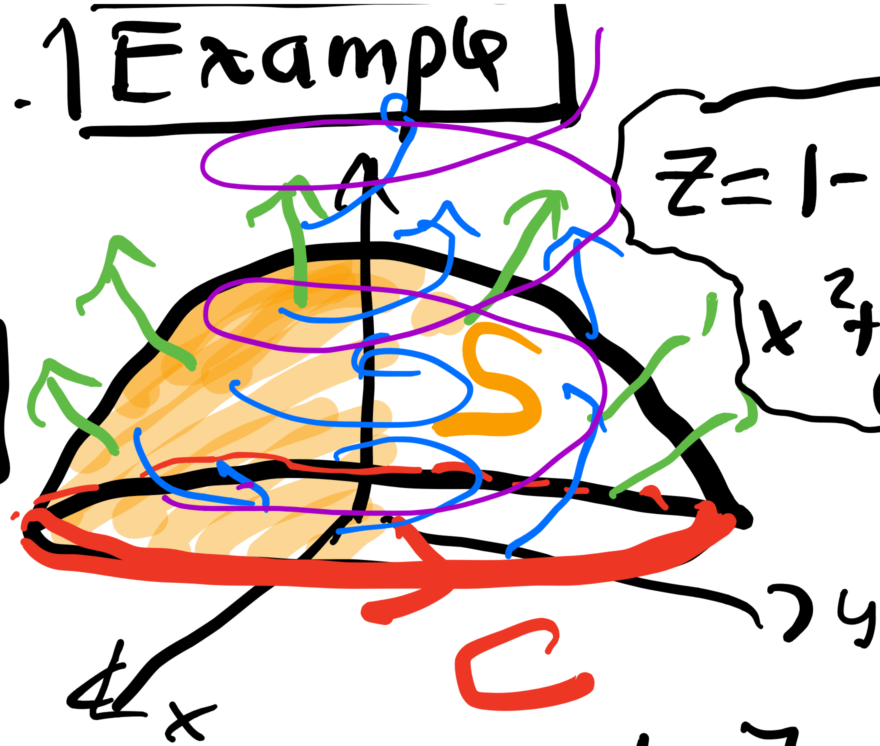
$$\boxed{\nabla f = \operatorname{grad} f}$$

- $\iint_S \operatorname{curl}(\vec{F})\, d\vec{S} = 0$ because $\operatorname{curl}(\operatorname{grad} f) = 0$

- $\oint_C \vec{F} \cdot d\vec{r} = 0$ because of the closed loop property (FTLI)

2.1 Example

$\vec{F} = \begin{bmatrix} -y \\ x \\ z^4 \end{bmatrix}$



$$\vec{r}(u,v) = \begin{bmatrix} u \\ v \\ 1-u^2-v^2 \end{bmatrix}$$

$$\underline{\vec{r}_u \times \vec{r}_v} = \begin{bmatrix} 1 \\ 0 \\ -2u \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -2v \end{bmatrix} =$$

"up" $$= \begin{bmatrix} 2u \\ 2v \\ 1 \end{bmatrix}$$

$$\vec{r}(t) = \begin{bmatrix} \cos t \\ \sin t \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$ is compatible with orient of S

Lets compute both sides!

$$\operatorname{Curl}(\vec{F}) = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} -y \\ x \\ z^4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\iint_{u^2+v^2\leq 1} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2u \\ 2v \\ 1 \end{bmatrix} du\,dv$$

$$= \iint_{u^2+v^2\leq 1} 2 \, du\,v$$

$$= 2 \iint_R 1 \, du\,dv$$

$$= 2 \operatorname{Area}(R) = \boxed{2\pi}$$

The line integral: $\vec{F} = \begin{bmatrix} -y \\ x \\ z^4 \end{bmatrix}$

$$\int_0^{2\pi} \begin{bmatrix}-\sin t\\ \cos t\\ 0^{4}\end{bmatrix} \cdot \begin{bmatrix}-\sin t\\ \cos t\\ 0\end{bmatrix} dt$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt = \int_0^{2\pi} 1 \, dt = \boxed{2\pi}$$

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③ **Complicated line integral**

$$\vec{r}(t) = \begin{bmatrix} 3\cos t \\ 0 \\ 3\sin t \end{bmatrix}$$

C :

$$\vec{F} = \begin{bmatrix} x^{11} + 2x^2 z + z \\ \cos(e^y) \\ 2xz^2 + \sin^5 z \end{bmatrix}$$

Find the $\int_C \vec{F} \cdot d\vec{r}$

Use Stokes theorem:

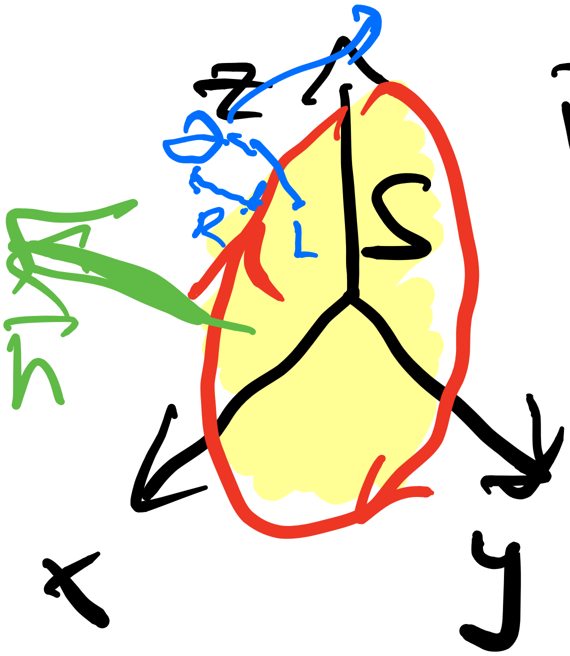
$$\text{curl}\,\vec{F} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} x^{11} + 2x^2 z + z \\ \cos e^y \\ 2xz^2 + \sin^5 z \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Use Stokes theorem

Epiphany

We have no surface!



$$\vec{r}(u,v) = \begin{bmatrix} u \\ 0 \\ v \end{bmatrix}$$

$$u^2+v^2 \le 9$$

$$\vec{r}_u \times \vec{r}_v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This is compatible

$$= \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \bar{n}$$

$$\iint_{u^2+v^2 \le 9} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} du\,dv$$

$$= \iint_{u^2+v^2 \le 9} (-1)\,du\,dv = \boxed{-\pi 9}$$

$$\boxed{-9\pi}$$

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④ **Computing complicated flux**

$x^2+y^2+z^2 \le 1$

$z \ge 0$

oriented upwards

$$\vec{F} = \left[-y+ze^{xyz},\ z,\ zx\right]$$

Find $\iint \operatorname{curl}(\vec{F}) \cdot d\vec{S}$

Use Stokes!

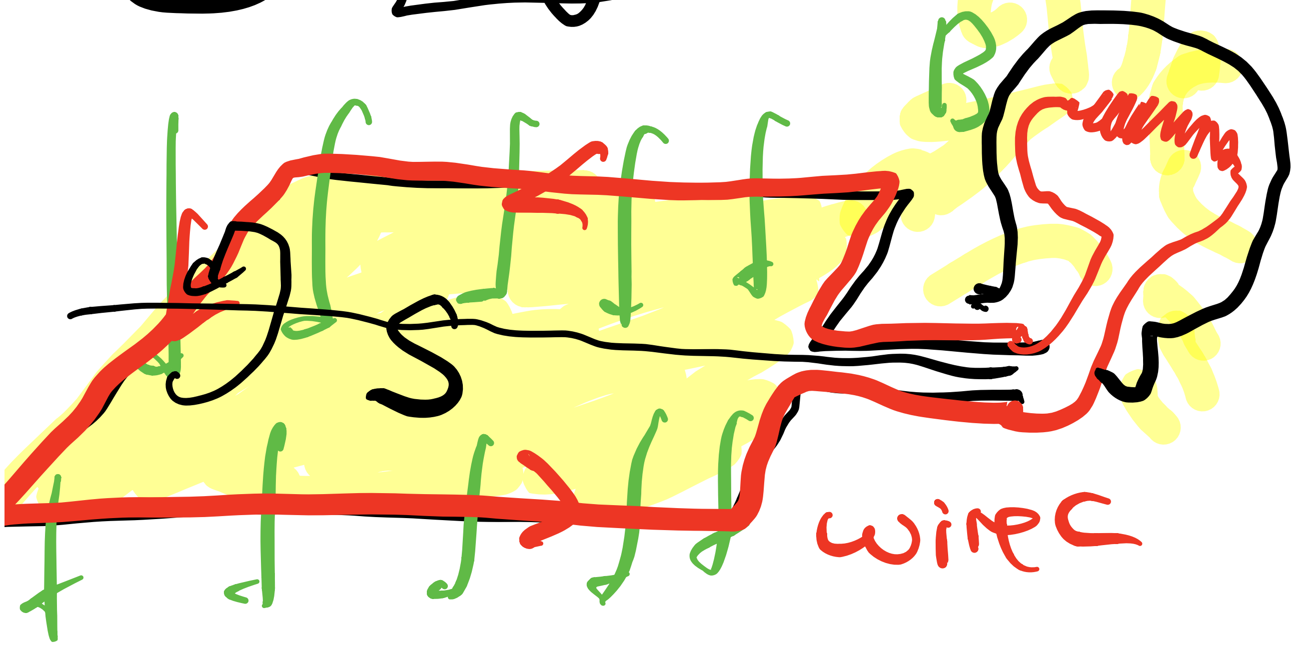
$$\int_C \vec{F}\cdot d\vec{r}$$

$$\vec{r}(t)=\begin{bmatrix}\cos t\\ \sin t\\ 0\end{bmatrix}=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, \quad 0\le t\le 2\pi$$

$$\int_0^{2\pi}\begin{bmatrix}-\sin t\\ 0\\ 0\end{bmatrix}\cdot\begin{bmatrix}-\sin(t)\\ \cos t\\ 0\end{bmatrix}dt$$

$$=\int_0^{2\pi}\underbrace{\sin^2 t}_{\frac{1-\cos 2t}{2}}\,dt=\boxed{\pi}$$

5) Dynamo



$$\iint_S \vec{B}\, d\vec{S}$$ magnetic flux

$$\frac{d}{dt}\iint_S \vec{B}\cdot d\vec{S}$$ change of flux

voltage

$$= \iint_S \frac{d}{dt}\vec{B}\cdot d\vec{S} = -\iint_S \mathrm{curl}(\vec{E})\cdot d\vec{S} \underset{\text{Stokes}}{=} \int_C \vec{E}\cdot d\vec{r} = V$$

Maxwell wrote

$$\boxed{\begin{aligned} \operatorname{div} E &= 4\pi\sigma \\ \operatorname{div} \vec{B} &= 0 \\ \operatorname{curl}(\vec{E}) &= -\frac{d}{dt}\vec{B} \\ \operatorname{curl}(B) &= \vec{j} + \frac{d}{dt}\vec{E} \end{aligned}}$$

charge

$\rightarrow$ light

electromagnetism.

in 4D: $(\vec{E}, \vec{B}) = F$

has six comp.

is curl of a 4D

vector field A

$$\boxed{\begin{aligned} dF &= 0 \\ d^*F &= j \end{aligned}}$$

in 4D

$$\operatorname{curl}(\vec{E}) = -\frac{d}{dt} B$$