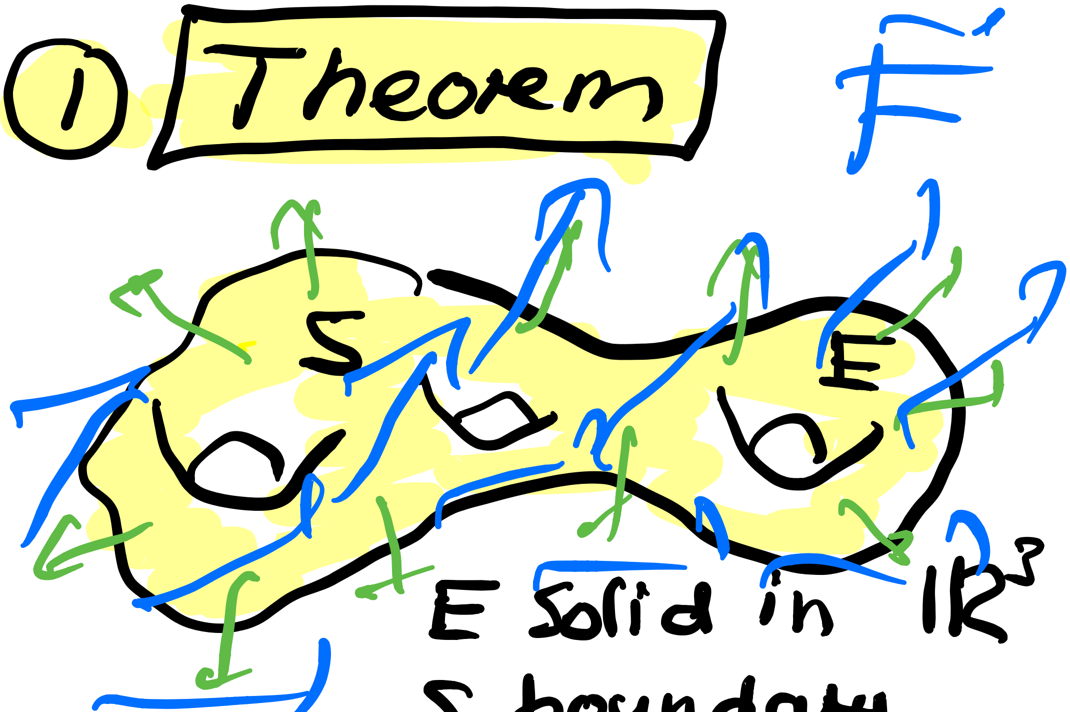
# Unit 24

# Divergence theorem

## ① Theorem




E solid in $\mathbb{R}^3$

S boundary surface = "skin"

oriented outwards.

**Theorem: Divergence theorem**

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$$

$\operatorname{div}\vec{F} = \operatorname{div}([P, Q, R])$

$= P_x + Q_y + R_z$ is

a <u>scalar field</u>.

measures expansion of $\vec{F}$

$\iiint_E \operatorname{div}\vec{F} \cdot dV$ is the total amount of field generated inside $E$.

$\iint_S \vec{F} \cdot d\vec{S}$ the amount of field leaving $E$ through the surface.

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<u>Example</u>:


$E$ unit ball

$S$ unit sphere

$\vec{F} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T$ $\operatorname{div}\vec{F} = 3$

$$\iiint_E \operatorname{div} F \, dV = 3 \operatorname{vol}(E)$$

$$= \frac{4\pi}{3} \cdot 3$$

$$= \boxed{4\pi}$$

Flux of $\bar{F}$ through $S$ is equal to the <u>surface area</u> (HW)

$$\boxed{4\pi}$$

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Intermission

5 min 10:05

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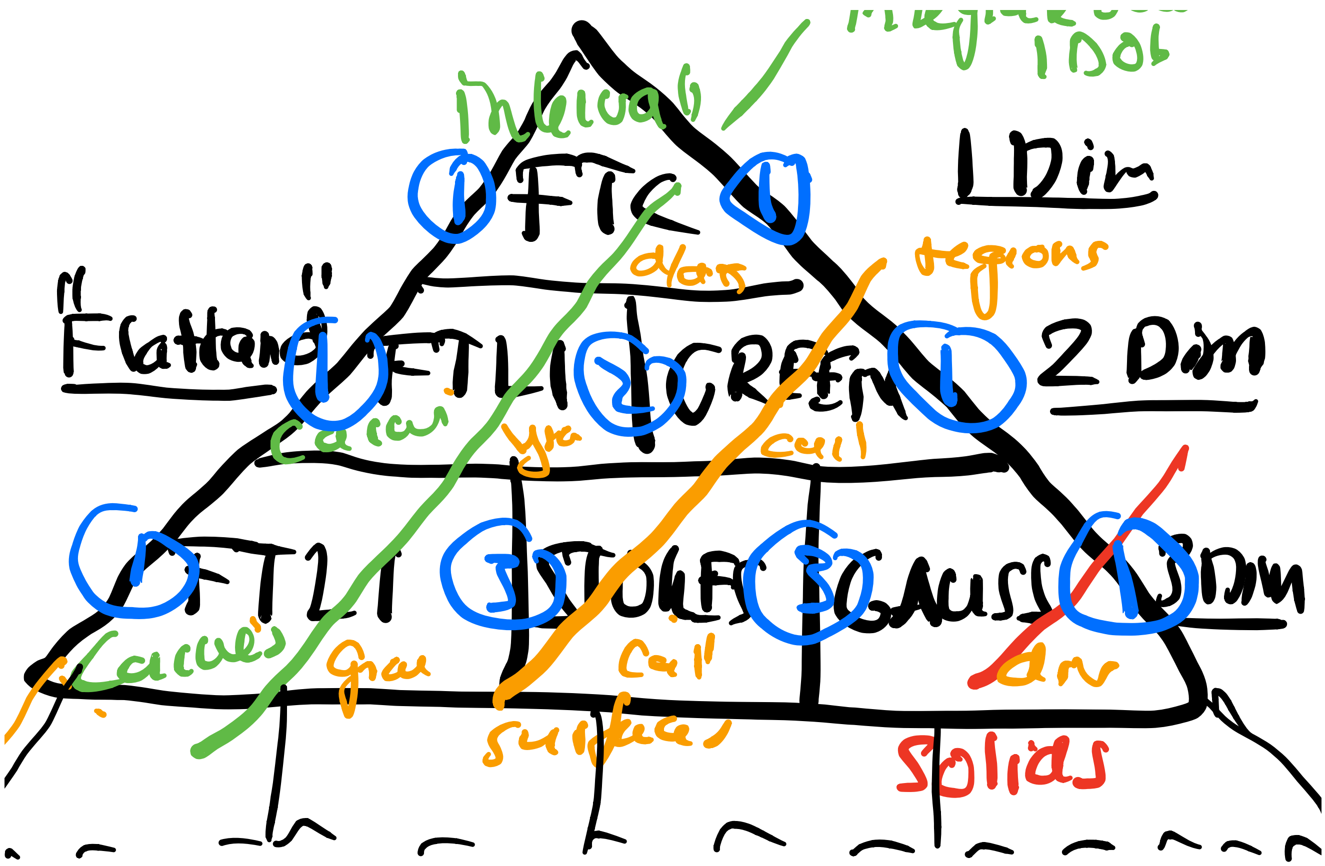
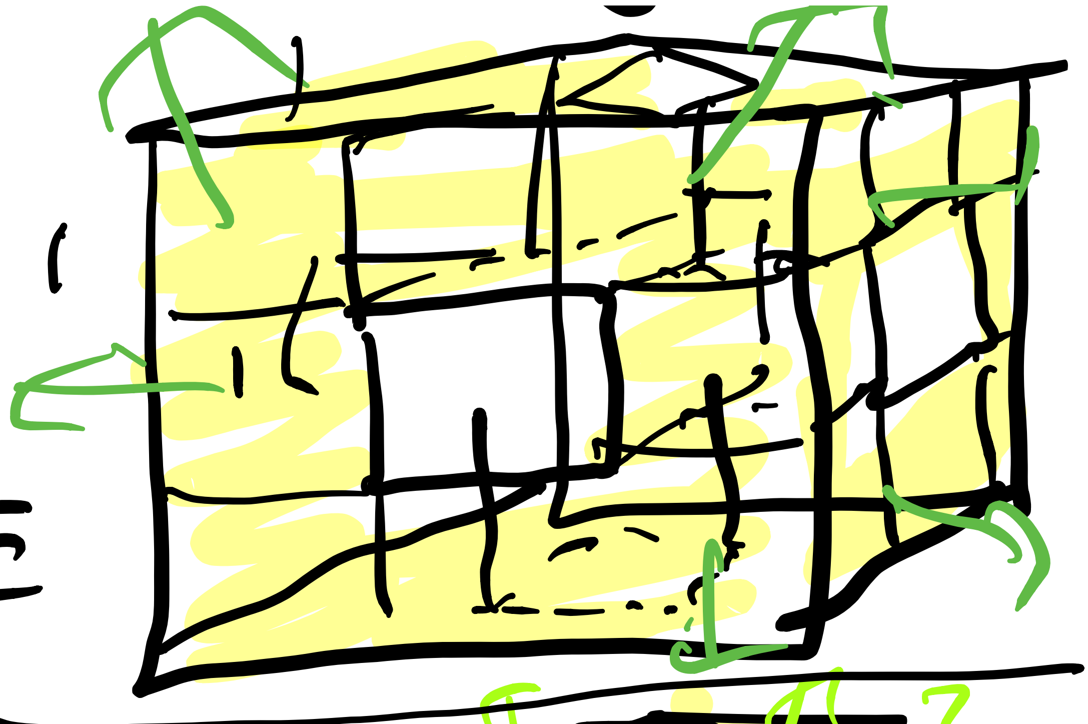
in break now




② **Complicated flux integrals**

$$\vec{F} = \begin{bmatrix} 5x + e^{yz} + \sin(e^{y}) \\ y + \sin(z) \\ xy\sin(e^{y}) \end{bmatrix}$$

E




Menger Sponge Fractal

**Problem**

Find the flux $\iint_S \vec{F} \cdot d\vec{S}$

S boundary of E oriented outwards

$\operatorname{div} \vec{F} = 6$

Divergence thm

$$\iiint_E \operatorname{div}(\vec{F})\, dV$$

$$= 6 \iiint_E 1\, dV$$

$$= 6 \operatorname{vol}(E)$$

$$= 6 \cdot 20 = \boxed{120}$$

# 3 complicated volumes

what is the volume of $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} \leq 1$

Use a vectorfield with $div(\vec{F}) = 1$

$\vec{F} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$, $\vec{F} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$

# Compute the flux

$$\vec{r}(\varphi,\theta) = \begin{bmatrix} 2\sin\varphi\cos\theta \\ 3\sin\varphi\sin\theta \\ 4\cos\varphi \end{bmatrix} \begin{matrix} x \\ y \\ z \end{matrix}$$

$$\vec{r}_\varphi \times \vec{r}_\theta = \sin\varphi \begin{bmatrix} 12\sin\varphi\cos\theta \\ 8\sin\varphi\sin\theta \\ 6\cos\varphi \end{bmatrix}$$

$$\int_0^{2\pi}\int_0^{\pi} \begin{bmatrix} 0 \\ 0 \\ 4\cos\varphi \end{bmatrix} \cdot \begin{bmatrix} 12\sin\varphi\cos\theta \\ 8\sin\varphi\sin\theta \\ 6\sin\varphi\cos\varphi \end{bmatrix} d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} 24\cos^2\varphi\sin\varphi \; d\varphi\, d\theta$$

$$\frac{-\cos^3\varphi}{3} \cdot 24 \Big|_0^{\pi} = 24\frac{2}{3}$$

$$\frac{4\pi}{3} \cdot 24$$

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$$\frac{4\pi}{3} abc$$

volume of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$

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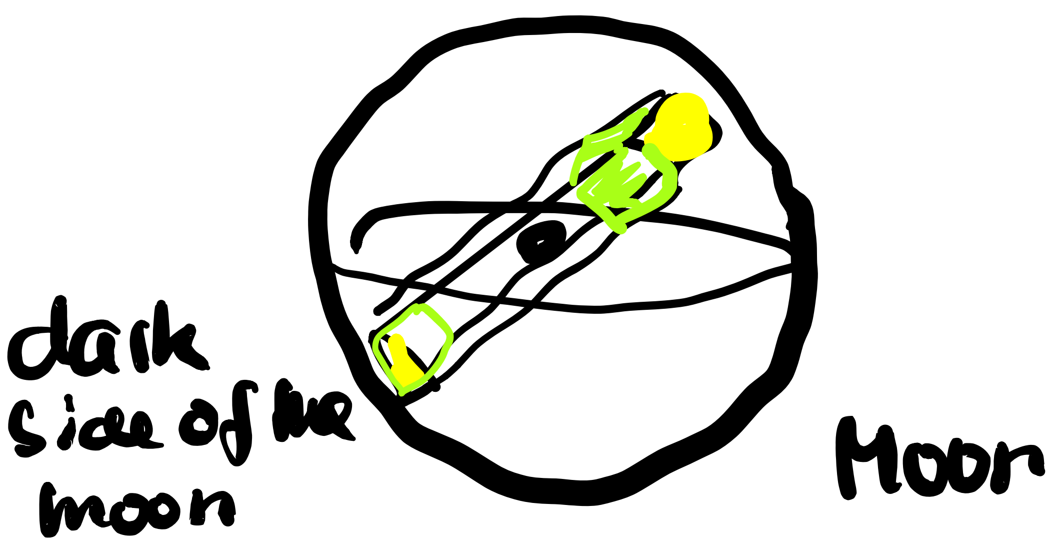
(5) Will park on the moon



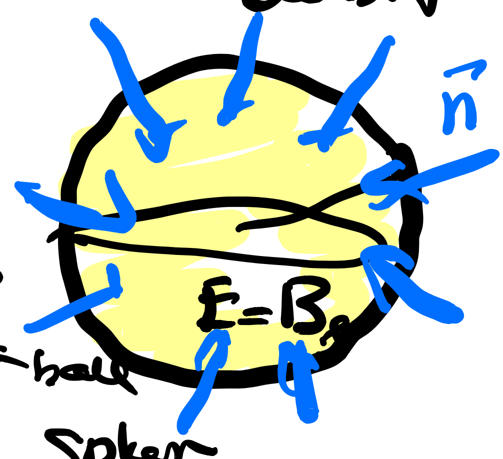
what is the gravity in distance $\rho$ from the center


Gauss described gravity using integral calc.

$$div(\vec{F}) = 4\pi\rho$$

gravit. field — mass density

Mass generates grav. fields




$B_\rho = \{x^2+y^2+z^2 \le \rho^2\}$ ← ball

$S_\rho = \{x^2+y^2+z^2 = \rho^2\}$ sphere

$$\vec{F} = F(\rho)\cdot\vec{n}$$

↑ unit vector

$$\iint_{S_\rho} \vec{F}\cdot d\vec{S} = F(\rho)\,4\pi\rho^2$$

Homework

surface area.

$$\|$$

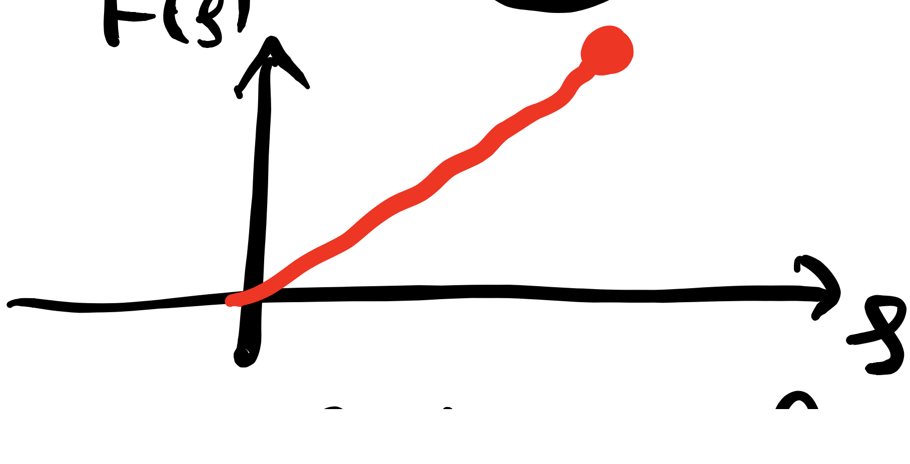
$$\iiint_{B_\rho} \operatorname{div}(\vec{F})\,dV = \iiint_{B_\rho} 4\pi\sigma\,dV$$

Gauss

$$= 4\pi\,\frac{4\pi}{3}\rho^3\sigma$$

$\Rightarrow$ solve for $F(\rho)$

$$F(\rho) = \frac{4\pi}{3}\sigma\,\rho = c\cdot\rho$$



If $\rho$ is $\geq$ $\delta$ moon

then $\iiint_{B_\rho} 4\pi\sigma \, dV$

$= 4\pi$ Mass of the moon

$= 4\pi M$

$\Longrightarrow$

$$F(\rho) = \frac{M}{\rho^2}$$

Newton law of gravity