7/9/2020 FIRST HOURLY Practice 1 **Maths 21a, O.Knill, Summer 2020**

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. At the end there is more space if needed.
- Do not detach pages from this exam packet or unstaple the packet.
- Except for problems 1-2 and 9, give details.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) T F

The plane $3x + 4y + z = 4$ intersects the y-axes in the point $(0, 1, 0)$.

Solution:

Yes, we put $x = z = 0$, we see $y = 1$.

2) T F

The Cauchy-Schwartz inequality implies that for two unit vectors $\vec{v}$ and $\vec{w}$ the dot product $\vec{v} \cdot \vec{w}$ is in the interval $[0, 1]$.

Solution:

It is in the symmetric interval $[-1, 1]$.

3) T F

Assume we have three points A, B, C such that $|\vec{AB}| = |\vec{AC}|$ then $A = C$.

Solution:

Take A the origin and B, C on the unit circle but different.

4) T F

The curve $\vec{r}(t) = [t^3, 2 + 2t^3, 3 + 3t^3]$ is a line.

Solution:

It is a parametrization of a line.

5) T F

If a smooth curve has curvature 1 everywhere, then its speed $|\vec{r}\ '(t)|$ is constant 1 everywhere.

Solution:

No, the curvature does not depend on the parametrization.

6) T F

The surface $x^2 - y + y^2 = z$ is a hyperbolic paraboloid.

Solution:

It is an elliptic paraboloid.

7) T **F** The angle between two vectors is always a number in $[0, \pi/2]$.

Solution:

It is a number in $[0\pi]$, not $[0, \pi/2]$.

8) **T** F If the velocity of a curve $\vec{r}(t)$ is a constant vector $\vec{v}$, then the curve is a line.

Solution:

Integrate to get $\vec{r}(t)$.

9) **T** F The lines $\vec{r}(t) = [0, t, -t]$ and $\vec{r}(t) = [t(1-t), t, 1-2t]$ do intersect.

Solution:

They intersect at time $t = 1$.

10) T **F** The formula given in spherical coordinates as $\rho = \rho\cos^2(\phi)$ defines a union of two planes.

Solution:

Multiply with ρ to get $x^2 + y^2 + z^2 = z^2$ which gives $x^2 + y^2 = 0$ which is the z-axis.

11) **T** F If $|\vec{u} \times \vec{v}| = 0$, then $\vec{u}$ and $\vec{v}$ are parallel in the sense that there exists a real number λ for which $\vec{u} = \lambda\vec{v}$.

Solution:

Yes, this means the angle between the two vectors is either 0 or π. We counted this right. The actual answer is actually no, as $\vec{v}$ can be zero and $\vec{w}$ not.

12) T F The curve given in polar coordinates as $\sin(\theta) + \cos(\theta) = r$ is a circle.

Solution:

Multiply with r to get $x + y = x^2 + y^2$ which is a circle.

13) T F The arc length a curve $\vec{r}(t)$ with $0 \leq t \leq 1$ is $|\int_0^1 \vec{r}\,'(t)\,dt|$.

Solution:

The absolute value has to be inside.

14) T F The surface parametrized by $\vec{r}(\phi, \theta) = [\phi, \phi^2+\theta^2, \theta]$ is an elliptic paraboloid.

Solution:

Yes, it reads $y = x^2 + z^2$.

15) T F There is a time t, when the velocity vector of $\vec{r}(t) = [\cos(t), \sin(t), t]$ is parallel to the vector $[0, 0, 1]^t$.

Solution:

We have $\vec{x}'(t)^2 + \vec{y}'(t)^2 = 1$ but we would need both $\vec{x}'(t)$ and $\vec{y}'(t)$ to be zero.

16) T F It is possible that the intersection of two ellipsoids is a hyperbola.

Solution:

The intersection is a bounded curve.

17) T **F** The function $f(x, y) = \sqrt{x^4 + y^3 + 1}$ has the entire plane as its domain.

Solution:

no, the expression under the square root can become negative.

18) T **F** The bi-normal vector $t \to \vec{B}(t)$ is a vector which always has a positive z component.

Solution:

The binormal vector can point into any direction.

19) **T** F The distance between two parallel planes is the distance of a point P in one plane to the other plane.

Solution:

Yes, it does not depend which point we take.

20) **T** F Given three vectors $\vec{u}, \vec{v}, \vec{w}$, then the vectors $\vec{u} + \vec{v}$, $\vec{u} + \vec{w}$, $\vec{v} - \vec{w}$ always are contained in a plane.

Solution:

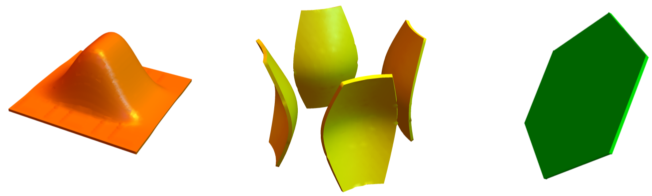
The difference of the first two is the third. So, if we parametrize the plane by taking a point on the plane and adding multiplies of the three vectors, we do not get the entire space but remain in the plane spanned by the first two vectors. This was maybe the hardest problem here. If you think about the three points obtained by taking the vectors from the origin to the point, then the three points are contained in a plane. But this was not the question.

Total

Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the contour surfaces $g(x, y, z) = 0$. Enter O, if there is no match.

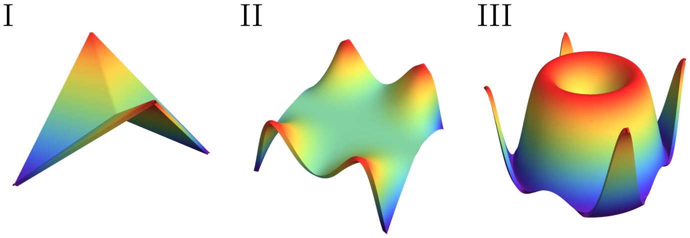
I II III



Function $g(x, y, z) =$	O,I,II or III
$x^2y^2 - z^2 = 1$	
$y^2 - z^8 = 1$	
$x + y + z = 0$	
$z - 2\exp(-x^2 - y^6) = 0$	

b) (2 points) Match the graphs of the functions $f(x, y)$. Enter O, if there is no match.

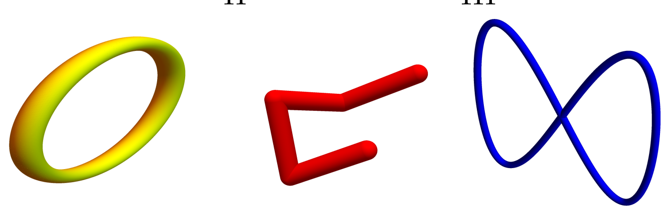
I II III



Function $f(x, y) =$	O,I,II or III
$\sin(x^2 + y^2)$	
$\|x - y\| - \|x + y\|$	
$y^2 \sin(x^2)$	
$\sin(x)$	

c) (2 points) Match the space curves with the parametrizations. Enter O, if there is no match.

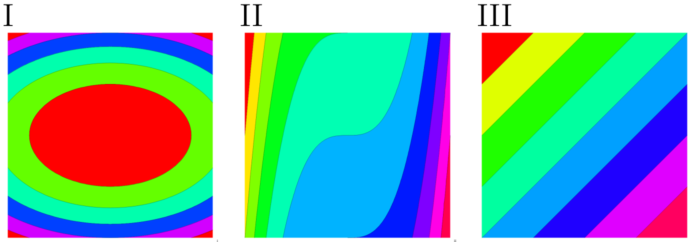
I II III



Parametrization $\vec{r}(t) =$	O, I,II or III
$[\cos(t), 0, \sin(2t)]$	
$[\|t\|, \|t - 1\|, \|\|t\| - 1\|]$	
$[0, \sin(2t), \cos(2t)]$	
$[t\cos(t), 0, t\sin(t)]$	

d) (2 points) Match the functions g with contour plots in the xy-plane. Enter O, if there is no match.

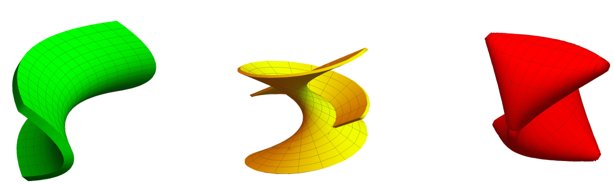
I II III



Function $g(x, y) =$	O, I,II or III
$2x^2 + 5y^2$	
$x - y$	
$x^3 - y$	
$\sin(x) + \sin(y)$	

e) (2 points) Match the surfaces. Enter O if there is no match.

I II III



Surface	O - III
$[u^2 - v^2, 2u, v]$	
$[u\cos[v], u\sin[v], v + u]$	
$[\cos(u)\cos(v), \cos(u)\sin(v), \sin(u)]$	
$[u^2, u^2 - v^2, v^2]$	

Solution:

II,0,III,1
III,I,II,0
III,II,I,0
I,III,0,0 (we actually had plotted $[\sin[u]\cos[v], \cos[u]\sin[v], \sin[u]]$. The answer [I,III,II,0] was counted true also "matching" indicated that there I,II,III,0 should all appear, and it was not obvious to see that the given parametrization is a sphere.)
I,II,III,0

Problem 3) (10 points)

Similarly as a pianist must practice etudes, or an athlete needs to push weights, a mathematician must practice basic computations. Our theme is build from the vectors

$$\vec{v} = \begin{bmatrix} 3 \\ 4 \\ 12 \end{bmatrix}, \ \vec{w} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} .$$

Please play as Allegro Sostenuto.

a) (2 points) What is the length of $\vec{v}$ and of $\vec{w}$?

b) (2 points) What is the dot product $\vec{v} \cdot \vec{w}$?

c) (2 points) What is the cross product $\vec{v} \times \vec{w}$?

d) (2 points) Find $\cos(\alpha)$ for the angle α between $\vec{v}, \vec{w}$.

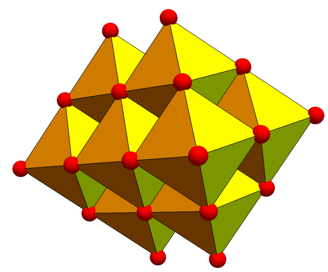
e) (2 points) What is the projection $\vec{P}_{\vec{w}}(\vec{v})$ of $\vec{v}$ onto $\vec{w}$?

Solution:

a) 13,3
b) 34
c) $[-4, 18, -5]$
d) 34/39
e) $(34/9)[2, 1, 2]$

Problem 4) (10 points)

Molybdates are compounds containing molybdenum and oxygen. An example is **decacoltanate** $M_{10}O_{28}$. Molybdates belong to the larger class of Polyoxometalates (POM) and are a hot spot in chemistry due to many applications, like pigments, batteries, semiconductors, photoactive materials etc. A decacoltanate can be visulized by 10 octahedra where each contains a Molybdenum atom. Each of the 10 octahedra is made of 8 triangles so that there are 80 triangles in this structure.



a) (3 points) Assume that one of the triangles has the vertex coordinates

$$A = (1, 2, 0), B = (0, 1, 2), C = (2, 0, 1) .$$

Find the area of the triangle ABC. Then find the total surface area of the structure.

b) (3 points) What is the equation $ax + by + cz = d$ for the plane through A, B, C?

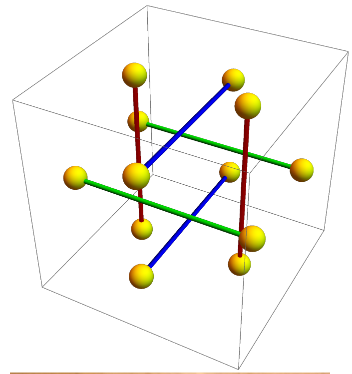
c) (2 points) Parametrize the line $\vec{r}(t)$ passing through A which is perpendicular to the triangle.

d) (2 points) What is the cos of the angle between the two vectors $\vec{AB}$ and $\vec{AC}$?

Solution:

a) Take the cross product of $\vec{AB}$ with $\vec{AC}$. It is $[3, 3, 3]$. Its length is $\sqrt{27}$. The area of one triangle is $\sqrt{27}/2$. The area of the structure is $40\sqrt{27}$.
b) $3x + 3y + 3z = 9$.
c) $\vec{r}(t) = [1, 2, 0] + t[3, 3, 3]$.
d) $\cos(\alpha) = 1/2$.

Problem 5) (10 points)

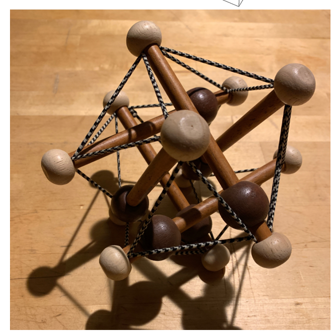


Oliver recently got an **icosahedron tensegrity model** that is made of 6 strouts. **Tensegrity** stands for tension and integrity and is a structural principle in architecture. Use a distance formula to get the distance between the line connecting A and B with

$$A = (-\phi, 1, 0), B = (\phi, 1, 0)$$

and the line connecting

$$C = (0, -\phi, 1), D = (0, \phi, 1) \ .$$



Here, $\phi = (1 + \sqrt{5})/2$ is the **golden ratio**. You can work with the letter ϕ in your computation and leave the result in terms of ϕ if you like.

Tensegrity is also used as a meditation mantra. Just repeat the sentence

“I remain stable even so I'm stressed and tense”

again and again and everything is fine.

Solution:

This is a standard distance problem. We have $\vec{n} = \vec{AB} \times \vec{CD} = [0, 0, 4\phi^2]$. Then $\vec{AC} = [\phi, -1-\phi, 1]$. The distance is $|\vec{n} \cdot \vec{AC}|/|\vec{n}| = 4\phi^2/(4\phi^2) = 1$.

Problem 6) (10 points)

During the 4th of July celebration near the **Charles river basin**, a rocket was observed to fly on the curve

$$\vec{r}(t) = \left[\ \frac{t^6}{6}, \frac{\sqrt{2}t^5}{5}, \frac{t^4}{4} \ \right] \ .$$

a) (2 points) Find the velocity of $\vec{r}(t)$ and the unit tangent vector.

b) (2 points) Is the unit tangent vector $\vec{T}$ defined at $t = 0$, the time the rocket lifts off? If yes, what is it, if no, why not?

c) (6 points) What is the arc length of the curve parametrized with $t \in [0, 1]$?

Solution:

a) $\vec{r}'(t) = [t^5, \sqrt{2}t^4, t^3] = t^3[t^2, \sqrt{2}t, 1]$.
b) $\vec{T} = \vec{r}'(t)/|\vec{r}'(t)|$ is at first only defined for $t \neq 0$, because $r'(0) = 0$. The unit tangent vector is $[0, 0, 1]$ at $t = 0$. So, we actually can continue T to $t = 0$. But we graded both answers correct.
c) $\int_0^1 t^3\sqrt{t^4 + 2t^2 + 1}\, dt = \int_0^1 t^3(t^2 + 1)\, dt = 1/6 + 1/4 = 5/12$.

Problem 7) (10 points)

The US soccer team won the world cup final. A memorable moment during the tournament was the tea drinking quip of **Alex Morgan** after scoring a goal against England in the semi-final. Alex hit the ball with a head shot. Assume the ball was hit at the position and velocity

$$\vec{r}(0) = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}, \vec{r}\,'(0) = \begin{bmatrix} 10 \\ 5 \\ 2 \end{bmatrix} .$$

and was subject to the force (acceleration)

$$\vec{r}\,''(t) = \begin{bmatrix} 0 \\ 1 \\ -10 \end{bmatrix} .$$

When and where does the ball hit the goal which is part of the surface $y = 21$?

Solution:

Integrate twice $\vec{r}(t) = [3 + 10t, t^2/2 + 5t + 2, -5t^2 + 2t + 5]$. The second coordinate solves $t^2 + 10t - 38 = 0$ which is zero at $t = 3\sqrt{7} - 5$. Plug in this to $\vec{r}(t)$ to get the point.

Problem 8) (10 points)

a) (4 points) The notation $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is due to Hamilton. Lets parse some expressions. Note that also in multivariable calculus, there is the PEMDAS rule, first parenthesis, exponentials, multiplication and divisions, then addition and subtraction.

The object	is vector	is scalar	is not defined
$(\vec{i}+1)\times(\vec{j}+1)$			
$(\vec{i}+\vec{k})\times(\vec{j}+\vec{k})$			
$\vec{i}+(\vec{i}\times\vec{j})+\vec{j}$			
$1+(\vec{i}\cdot\vec{j}+\vec{k})+1$			

By the way, there is a lot of more room for arithmetic here. You can ask yourself for example, what is $\vec{i}^{\vec{i}}$. There is mathematics which can make sense of this and assign a value like $e^{-\pi/2}$. But that needs the world of quaternions.

b) (4 points) Which expressions are independent of the curve parametrization?

Expression	parameter independent	parameter dependent
curvature of a curve at a point		
the arc length of a curve		
the velocity of a curve		
the unit tangent vector of a curve		

c) (1 point)

Write down the Cauchy-Schwarz inequality! Remember that we had proven that on the first day.

d) (1 point)

Write down a possible formula for the curvature of a curve $\vec{r}(t)$.

Solution:

a) 1 and 4 are not defined, 2 and 3 are vectors.
b) curvature and arc length are parameter independent, the others not. Marking the unit tangent vector to be parameter independent was not marked wrong as it is right if we would insist on having the same orientation for the curve. Technically it depends on the orientation. Take the circle and turn in one way or the other. The reason for accepting the other was that one could mean 'independent of the parametrization' as 'independent of parametrization of the same oriented curve'.

Problem 9) (10 points) No justifications are needed.

We have just had a few great hot summer days. While working for the exam, we enjoyed a **cool iced macchiato** with cream and a cherry on top. There are various ingredients. The cup is part of a cone, the bottom is a disc, the straw is a cylinder, the cream is a tube, the lemon slice is a graph of a function. Please complete the parametrizations. We don't ask you to parametrize the cream. It will be up to you to think about the next time you enjoy a nice caffè mocha.

a) (2 points) The **cup surface** is $x^2 + y^2 = z^2/25$.

$\vec{r}(r,\theta) = [\quad, \quad, \quad]$

b) (2 points) The **waffle lemon slice** $x = 4 - \sin(yz/5)$.

$\vec{r}(y,z) = [\quad, \quad, \quad]$

c) (2 points) The **cherry** $x^2 + y^2 + (z-14)^2 = 1$.

$\vec{r}(\theta,\phi) = [\quad, \quad, \quad]$.

d) (2 points) The **cookie straw** $(x-5)^2 + (z-16)^2 = 1$

$\vec{r}(y,\theta) = [\quad, \quad, \quad]$.

e) (2 points) The **table plane** $z = 15$ containing the **coaster** $z = 15$.

$\vec{r}(x,y) = [\quad, \quad, \quad]$

Solution:

a) $[r\cos(\theta), r\sin(\theta), 5r]$.
b) $[4 - \sin(yz/5), y, z]$.
c) $[\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi) + 14]$.
d) $[\cos(\theta) + 5, y, \sin(\theta) + 16]$.
e) $[x, y, 15]$.