7/9/2020 FIRST HOURLY Practice 2 **Maths 21a, O.Knill, Summer 2020**

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. At the end there is more space if needed.
- Do not detach pages from this exam packet or unstaple the packet.
- Except for problems 1-2 and 9, give details.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) T F The plane $3x + 4y + z = 4$ and the line $\vec{r}(t) = [3t, 4t, t]$ intersect in a point.

2) T F The Cauchy-Schwartz inequality is the following formula: $|\vec{v} \cdot \vec{w}| \geq |\vec{v}||\vec{w}|$ for all $\vec{v}, \vec{w}$.

3) T F If the vector $\vec{PQ}$ has length 0, then $P = Q$.

4) T F The curve $\vec{r}(t) = [1 + 7t, 0, t]$ is a line contained in the xz-plane.

5) T F If a curve has constant speed 1, then the curvature is the magnitude of the acceleration.

6) T F The surface $x^2 + y^2 + 10y = z^2$ is a cone.

7) T F If the angle between two vectors $\vec{v}, \vec{w}$ is $\pi/2$, then the angle between $\vec{w}$ and $\vec{v}$ is $-\pi/2$

8) T F If the velocity of a curve $\vec{r}(t)$ is zero at all t, then $\vec{r}(t) = \vec{r}(0)$ for all t.

9) T F The lines $\vec{r}(t) = [-3t, 5t, 7]$ and $\vec{r}(t) = [5t, 3t, 0]$ intersect perpendicularly.

10) T F The formula $\rho = \rho \sin(\phi)$ defines a plane.

11) T F Given two vectors $\vec{u}, \vec{v}$ and $\vec{w}$, then $|(\vec{u} \times \vec{v})| = |\vec{u}||\vec{v}| - |\vec{u} \cdot \vec{v}|$.

12) T F The surface given in spherical coordinates as $\cos(\theta) = \sin(\theta)$ is a plane.

13) T F The arc length of the space curve $[t, \sin(t), 0]$ from $t = 0$ to $t = 2\pi$ is the same than the arc length of the planar curve $[t, \sin(t)]$ from $t = 0$ to $t = 2\pi$.

14) T F The surface parametrized by $\vec{r}(u, v) = [u^7, v^7, 0]$ is a plane.

15) T F The length of the cross product of two unit vectors can have length 2.

16) T F It is possible that the intersection of an ellipsoid with a plane is a parabola.

17) T F The function $f(x, y) = \sin(\sin(x))/x$ is continuous at $(0, 0)$ in the sense we have defined it: we can assign a value $f(0, 0)$ so that the extended function is continuous.

18) T F Given a curve parametrization $\vec{r}(t)$. Then, the binormal vector $t \to \vec{B}(t)$ in the TNB frame is a continuous vector valued function.

19) T F The distance between two lines L, M is the distance of a point $P \in L$ to the line M.

20) T F The vector $\vec{v} + \vec{w}$ is perpendicular to $\vec{v} - \vec{w}$ if both $\vec{v}$ and $\vec{w}$ are unit vectors.

Total

Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the contour surfaces $g(x,y,z)=0$. Enter O, if there is no match.

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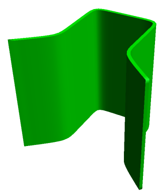

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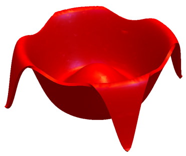

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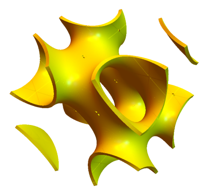

Function $g(x,y,z)=$	O-III
$\cos(x+y)+\cos(y+z)+\cos(z+x)=0$	
$y^9+x^8-1=0$	
$x+y+z=0$	
$z+\sin(x^2+y^2)=0$	

b) (2 points) Match the graphs of the functions $f(x,y)$. Enter O, if there is no match.

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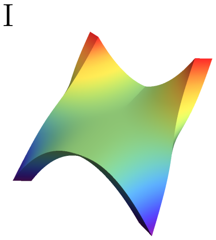

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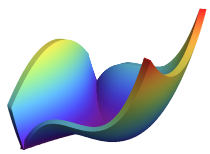

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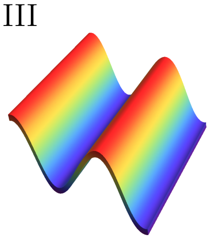

Function $f(x,y)=$	O,I,II or III
x^2y	
$y+x$	
$\lvert x^3+y^2\rvert$	
$\sin(x)$	

c) (2 points) Match the space curves with the parametrizations. Enter O, if there is no match.

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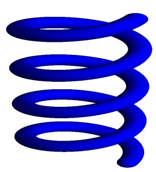

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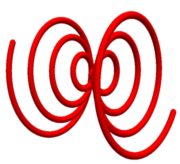

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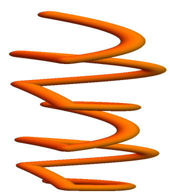

Parametrization $\vec{r}(t)=$	O, I,II or III
$[t\cos(4t), t\sin(4t), t]$	
$[\cos(4t), \sin(4t), t]$	
$[\sin(t), \cos(t)), \sin(2t)]$	
$[\lvert\sin(3t)\rvert, \lvert\cos(2t)\rvert, t]$	

d) (2 points) Match the functions g with contour plots in the xy-plane. Enter O, if there is no match.

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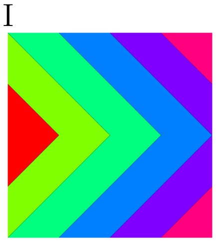

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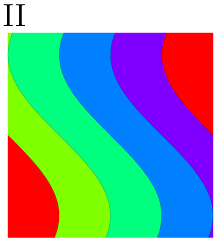

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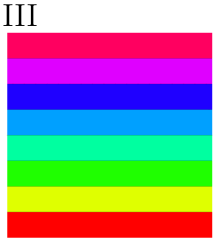

Function $g(x,y)=$	O, I,II or III
$x+\lvert y\rvert$	
x^3	
$x+\sin(y)$	
e^y	

e) (2 points) Match the surfaces. Enter O if there is no match.

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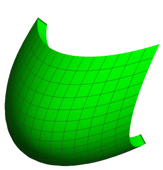

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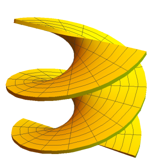

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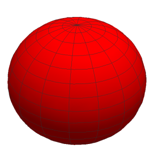
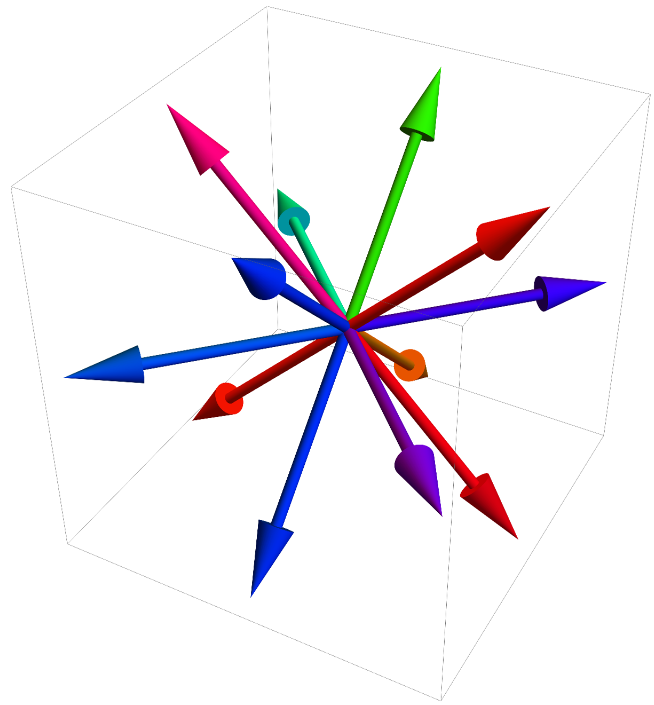


Surface	O,I,II or III
$x^2+y^2+z^2=1$	
$y=x^2+z^2$	
$y=x^2-z^2$	
$y=x\tan(z)$	

Problem 3) (10 points)



A **root system** is a collection of vectors in $\mathbf{R}^3$ which are not all part of a common plane, and have the property that if $\vec{v}, \vec{w}$ are roots, then $s = 2\frac{v \cdot w}{w \cdot w}$ is an integer and $-\vec{v}$ and $\vec{v} - s\vec{w}$ is in the root system. The A_3 root system consists of 12 vectors. Four of them, $[1,-1,0], [1,1,0], [-1,1,0], [-1,-1,0]$ are in the xy plane. Four analog vectors are in the yz plane, the rest in the xz plane. You see the 12 vectors displayed to the right. One can see quickly from Cauchy-Schwarz that in general only the angles $0, \pi/6, \pi/4, \pi/3, \pi/2, 4\pi/6, 3\pi/4, 5\pi/6, \pi$ can occur between two vectors in a root system. The classification of root systems is known in arbitrary dimensions and given by the list $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_4$. They classify Lie Algebras.

Pick $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

a) (3 points) Find the angle between $\vec{v}$ and $\vec{w}$.

b) (3 points) Find the number

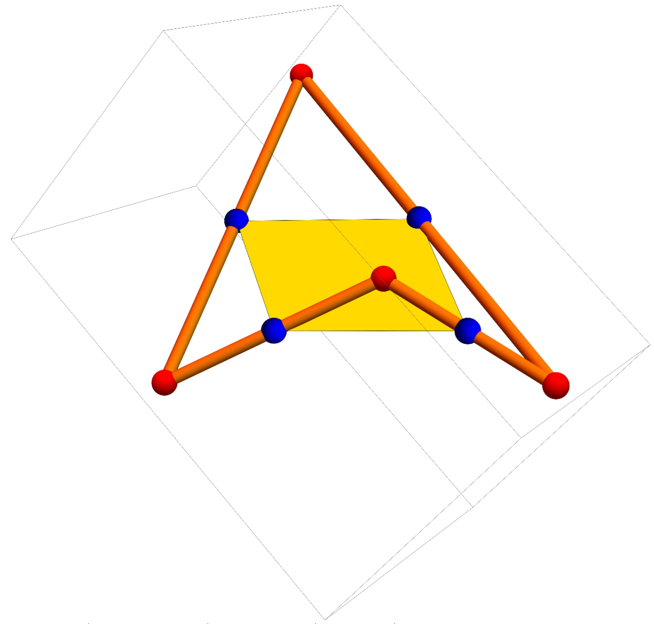
$$s = 2\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} .$$

As indicated in the story, this has to be an integer.

c) (4 points) The vector $\vec{u} = \vec{v} - s\vec{w}$ is the vector $\vec{v}$ reflected at the line spanned by $\vec{w}$. Compute this vector $\vec{u}$.

Problem 4) (10 points)

Varignon's theorem assures that the midpoints of a quadrilateral always define a parallelogram (possibly degenerate to a line segment or a point). Formally, if A, B, C, D are the four points of the quadrilateral in space, then $U = (A+B)/2, V = (B+C)/2, W = (C+D)/2$ and $Q = (D+A)/2$ always form a parallelogram if not contained in a common line. You can verify that $\vec{UV} = \vec{QW}$ and $\vec{VW} = \vec{UQ}$ in general but you do not need to. In this problem we want you to check a particular case.



We are given the four points $A = (3,5,7)$, $B = (1,1,1)$, $C = (-5,10,7)$, $D = (9,5,9)$.

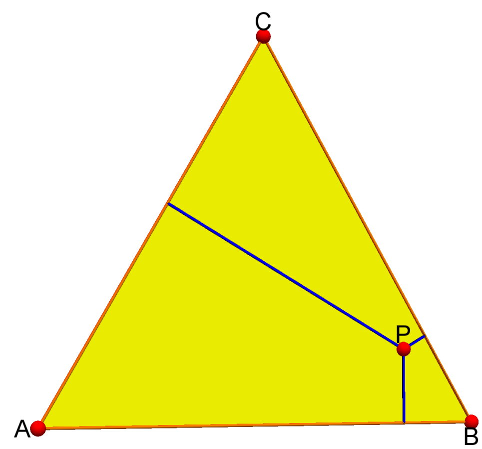
a) (4 points) Find $U = (A+B)/2$, $V = (B+C)/2$, $W = (C+D)/2$ and $Q = (D+A)/2$.

b) (2 points) Verify that $\vec{UV} = \vec{QW}$ and $\vec{VW} = \vec{UQ}$ in the particular example.

c) (4 points) What is the area of the parallellogram $UVWQ$?

Problem 5) (10 points)

A nice theorem related to the topic of **Barycentric coordinates** in a triangle assures that the sum of distances to the sides of equilateral triangle is the height of the triangle.
Let us check the in the case of the triangle $A = (-3, 0, 0), B = (3, 0, 0), C = (0, 3\sqrt{3}, 0)$.



a) (3 points) Take the point $P = (2, 1, 0)$. What is the distance to the line

$$\vec{r}(t) = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} ?$$

b) (3 points) Find the distance of $P = (2, 1, 0)$ to the line

$$\vec{r}(t) = \begin{bmatrix} 3 - t \\ t\sqrt{3} \\ 0 \end{bmatrix} ?$$

c) (3 points) Find the distance of P to the line

$$\vec{r}(t) = \begin{bmatrix} -3 + t \\ t\sqrt{3} \\ 0 \end{bmatrix} ?$$

d) (1 point) Check that the three answers in a),b) and c) add up to the height $3\sqrt{3}$ of the triangle.

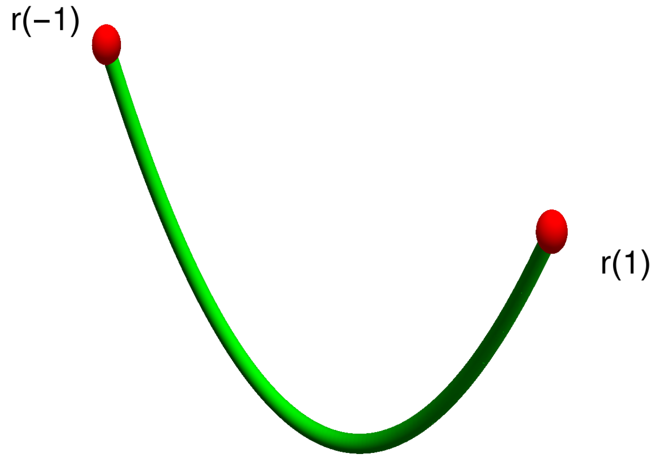
Problem 6) (10 points)

Find the arc length of the **catenary curve** or "chain curve".

$$\vec{r}(t) = \begin{bmatrix} 2t \\ 3 \\ e^t + e^{-t} \end{bmatrix},$$

where the parameter t ranges from -1 to 1.

(Some history tidbit: **Galileo Galilei** thought that a chain curve is a parabola. This was later disproved by Jungius. When we rotate the chain curve we get the **catenoid** $r = e^z + e^{-z}$. Euler proved that it is surface of minimal surface area.)

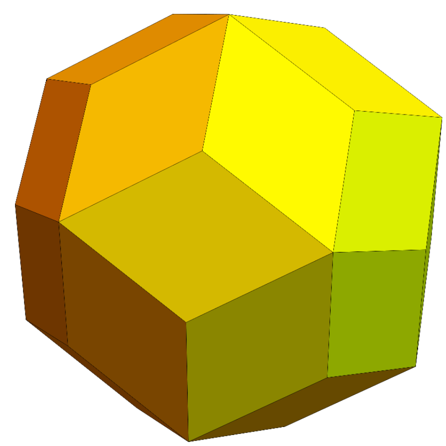


Problem 7) (10 points)

The **triacontahedron** is a thirty-faced convex polyhedron. It belongs to the **Catalan solids**. It has 30 congruent rhombic faces, where each is a **golden rhombus**. Let us assume that the coordinates of one of these rhombi is

$A = (-\phi, 0, 0), B = (0, -1, 0), C = (\phi, 0, 0), D = (0, 1, 0)$

where $\phi = (\sqrt{5} + 1)/2$ is the **golden ratio.**



a) (8 points) Find the area of the golden rhombus $ABCD$. No "smarty pants" solution please: we want you to derive the expression using one of the products defined for vectors, even if the answer should be "obvious" to you.

b) (2 points) What is the surface area of the triacontrahedron in which $ABCD$ is one of the faces?

Problem 8) (10 points)

We have some nice sports events this summer, the Tour de France in cycling, the World cup 2018 in soccer and the Whimbleton tennis tournament. **Roger Federer**, the Swiss tennis player does there currently quite well and advanced to the quarter final. He hits a low ball with initial velocity

$$\vec{r}\,'(0) = \begin{bmatrix} 70 \\ 0 \\ 4 \end{bmatrix}$$

from the point

$$\vec{r}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} .$$

Assume the ball is subject to a force

$$\vec{r}\,''(t) = \begin{bmatrix} 0 \\ \sin(t) \\ -10 \end{bmatrix}$$

which is combination of a time dependent side wind and a gravitational force.

a) (6 points) Find the trajectory $\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$.

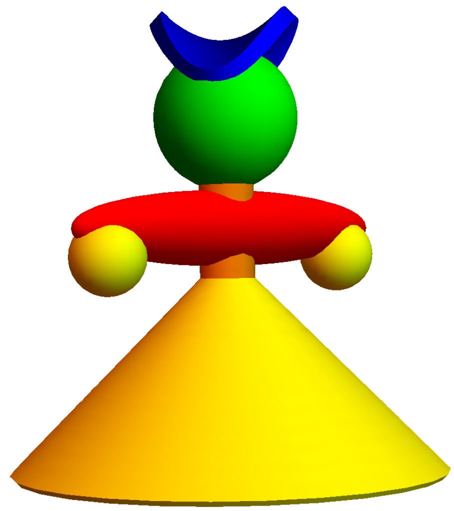
b) (2 points) At which positive time t does the ball hit the grass $z = 0$?

c) (2 points) Where does the tennis ball hit the grass $z = 0$?

Problem 9) (10 points)

We produce some candy figures for a game called **candy chess**. Unlike with other chess games, the figures are edible. If you catch an opponents figure in that game, you have to eat it! For us this is nice because it allows **Math-Candy.com** to sell lots of games of this kind.

The **queen** in our candy game is made of different parts. Parametrize them using the variables provided. These are the standard variables we have seen.



a) (2 points) The head is $x^2 + y^2 + (z - 10)^2 = 9$.

$\vec{r}(\theta, \phi) = [\quad , \quad , \quad]$.

b) (2 points) The neck is part of the cylinder $x^2 + y^2 = 1$.

$\vec{r}(\theta, z) = [\quad , \quad , \quad]$.

c) (2 points) The crown is part of the graph $z = \sin(xy) + 13$.

$\vec{r}(x, y) = [\quad , \quad , \quad]$.

d) (2 points) The skirt is part of $x^2 + y^2 = (z + 1)^2$.

$\vec{r}(\theta, z) = [\quad , \quad , \quad]$.

e) (2 points) The shoulder part is given by $x^2 + y^2/25 + z^2 = 1$.

$\vec{r}(\phi, \theta) = [\quad , \quad , \quad]$.