7/23/2020 SECOND HOURLY PRACTICE V Maths 21a, O.Knill, Summer 2020

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

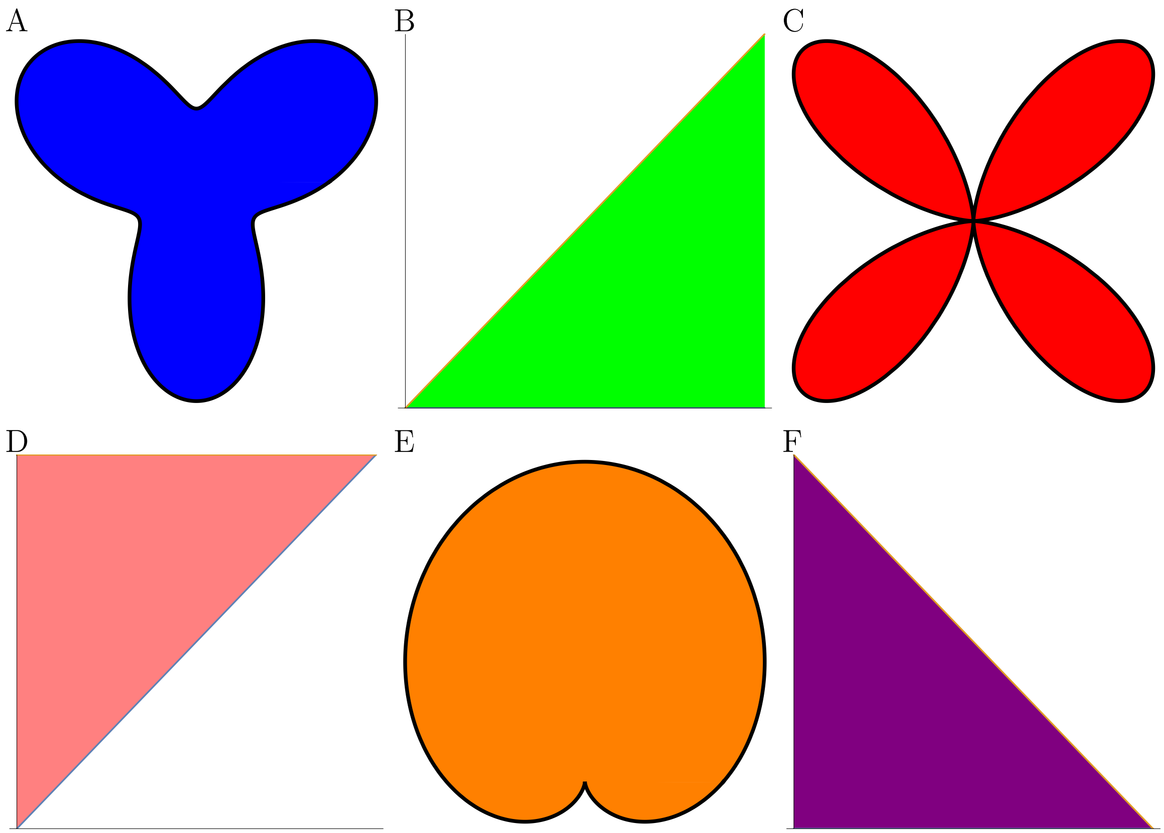
Problem 1) True/False questions (20 points). No justifications needed.

Mark for each of the 20 questions the correct letter. No justifications are needed.

1)	T	F	If $x^2 + 2y^2 + 3z^2 = 1$ describes the surface of an ellipsoid, then the gradient ∇f points outwards.
2)	T	F	The chain rule tells that $\frac{d}{dt}\vec{r}(f(t)) = \nabla \vec{r}(f(t)) \cdot \vec{f}\,'(t)$ for any function $f(x, y, z)$ and any curve $\vec{r}(t)$.
3)	T	F	The function $f(x, y) = x^4 - y^4$ has infinitely many critical points.
4)	T	F	The point $(0, 1)$ is a maximum of $f(x, y) = y$ under the constraint $g(x, y) = x^2 + y^2 = 1$.
5)	T	F	Every linear function $u(x, y) = ax + by + c$ solves the partial differential equation $u_{xx}(x, y) = u_{yy}(x, y)$.
6)	T	F	Let $f(x, y) = x^3 y$. At every point (x, y) there is a direction v for which $D_v f(x, y) = 0$.
7)	T	F	If $f_{xy} = f_{yx}$ then $f(x, y) = xy$.
8)	T	F	$g(x, y) = \int_0^x \int_0^y f(s, t)\, dsdt$ satisfies $g_{xy} = f(x, y)$.
9)	T	F	If $\vec{r}(u, v)$ is a parametrization of the level surface $f(x, y, z) = c$, then $\nabla f(\vec{r}(u, v)) \cdot \vec{r}_v(u, v) = 0$.
10)	T	F	The length of the gradient $\nabla f(0, 0)$ is the maximal directional derivative $\lvert D_{\vec{v}} f(0, 0)\rvert$ among all unit vectors $\vec{v}$.
11)	T	F	Given a parametrization $\vec{r}(t)$ of a curve and a function $f(x, y)$ we have $\frac{d}{dt} f(\vec{r}(2t)) = 2\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ at $t = 0$.
12)	T	F	$\int_0^{2\pi} \int_0^2 r\, drd\theta = 4\pi$.
13)	T	F	If $u(t, x)$ solves both the heat and wave equation, then $u_t = c\, u_{tt}$ for some constant c.
14)	T	F	If the Lagrange multiplier λ at a solution to a Lagrange problem is negative then this point is neither a maximum nor a minimum.
15)	T	F	The equation $f_{xy} f_{xx} f_{yy} = 1$ is an example of a partial differential equation.
16)	T	F	If the discriminant D of $f(x, y)$ is positive at $(0, 0)$ then $\lvert\nabla f(0, 0)\rvert > 0$.
17)	T	F	If $f(x, y)$ is a continuous function then $\int_7^9 \int_5^7 f(x, y)\ dxdy = \int_5^7 \int_7^9 f(y, x)\, dxdy$.
18)	T	F	The point $(-3, 1)$ is a critical point of $f(x, y) = x^2 + 3y^2$.
19)	T	F	If f has the critical point $(0, 0)$, then f_x has the critical point $(0, 0)$.
20)	T	F	$\int\int_R \sqrt{f_x^2 + f_y^2}\ dxdy$ is the surface area of the graph $z = f(x, y) = x^3 + y^3$ defined over the region R.

Problem 2) (10 points) No justifications are needed

a) (6 points) Match the following regions with their area computation.



Enter A-F	Area Integral
	$\int_0^1 \int_y^1 1 \; dxdy$
	$\int_0^{2\pi} \int_0^{\sin^2(\theta)\cos^2(\theta)} 1 \; rdrd\theta$
	$\int_0^{2\pi} \int_0^{2+\sin(3\theta)} 1 \; rdrd\theta$
	$\int_0^1 \int_0^{1-x} 1 \; dydx$
	$\int_0^{2\pi} \int_0^{1+\sin(\theta)} 1 \; rdrd\theta$
	$\int_0^1 \int_0^y 1 \; dxdy$

b) (4 points) Match the following partial differential equations:

Fill in 1-4	Name
	Black Scholes
	Laplace
	Burgers
	Heat

Equation Number	PDE
1	$f_t = f_{xx}$
2	$f_t = f - xf_x - x^2 f_{xx}$
3	$f_t = ff_x - f_{xx}$
4	$f_{tt} + f_{xx} = 0$

Problem 3) (10 points) (No justifications are needed.)

a) (5 points) Each answer is a number:

i) If $|\nabla f| = 1$ and $\vec{v} = \nabla f$, then $D_{\vec{v}} f =$ [ ]

ii) The z component of the gradient ∇g of $g(x, y, z) = f(x, y) - z$ is [ ].

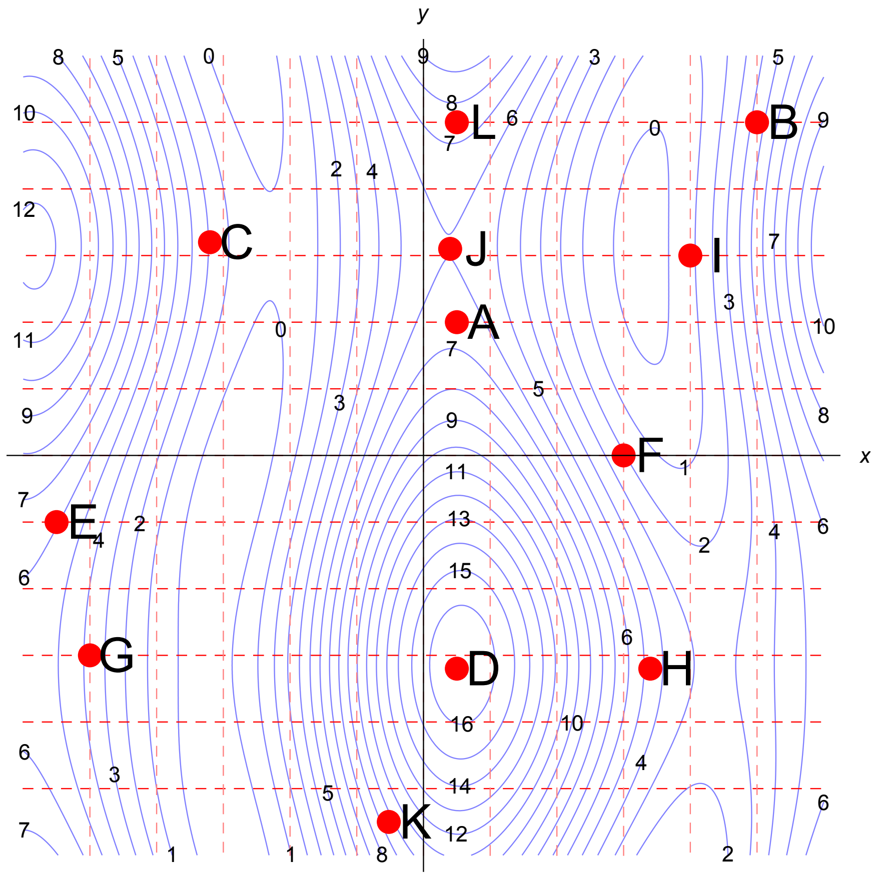
(iii) The directional derivative in a direction parallel to the level set is [ ].

(iv) If $\vec{r}(t)$ is a curve on $\{f = 1\}$, then $\frac{d}{dt} f(\vec{r}(t))$ is [ ]

(v) The angle between the normal vector $\vec{n}$ to $L(x, y) = z$ at a point (x_0, y_0) and the gradient of f at (x_0, y_0) is [ ]

b) (5 points) In each part, pick the correct point in $A - L$.

	Choose one A-H
A saddle point	
A local maximum	
A point where $\|\nabla f\|$ maximal	
A point where $f_x = 0$ and $f_y > 0$	
A point where $f_y = 0$ and $f_x > 0$	



Problem 4) (10 points)

Between Harvard and MIT, there is a building with octagonal cross section. The living area is $f(a,b) = a^2 + 2b^2$, while the illuminated window area is $g(a,b) = 4ab$. Solve with Lagrange:

a) (5 points) Find (a,b) minimizing f under the constraint $g = 4$.

b) (5 points) Find (a,b) maximizing g under the constraint $f = \sqrt{8}$.

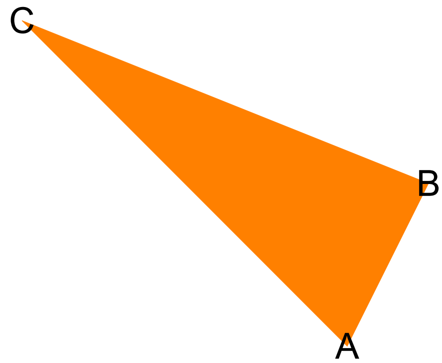
Of course, you should get the same solution $a > 0$ and $b > 0$ in both problems.

Problem 5) (10 points)

For every (x, y) we can define a triangle with vertices $(0,0), (x,y), (y-6x, 2x^2)$. Its area is

$$f(x,y) = 2x^3 - 6xy - y^2 \ .$$

Classify all critical points of f and in particular determine the values (x, y) for which the area is maximal.

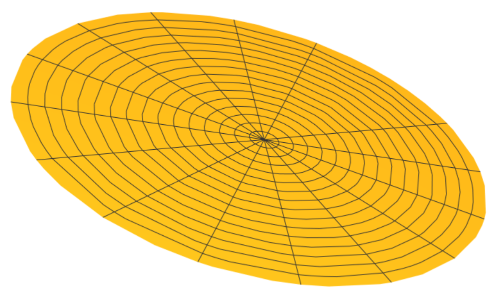


Problem 6) (10 points)

Find the surface area of the surface parametrized by

$$\vec{r}(u,v) = [u+v, u-v, u+v] \ ,$$

where u, v is in the unit disc $u^2 + v^2 \leq 1$.



Problem 7) (10 points)

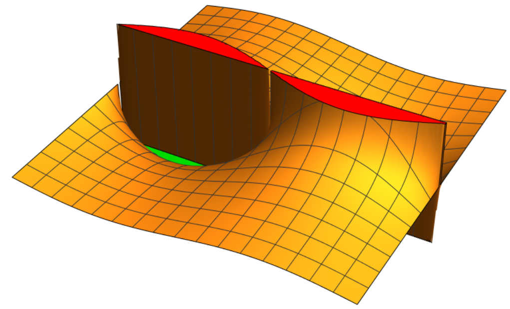
Estimate

$$\frac{\sin(\pi + 0.001)}{(2.0001)} + 2.0001$$

by linearizing

$$f(x, y) = \frac{\sin(x)}{y} + y$$

at $(\pi, 2)$.

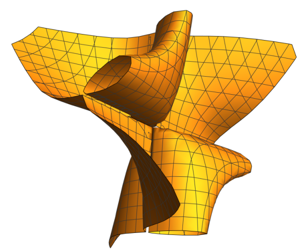


Problem 8) (10 points)

a) (5 points) Find the tangent plane to the surface

$$x^3 y^3 + z^5 - 2xyz = 0$$

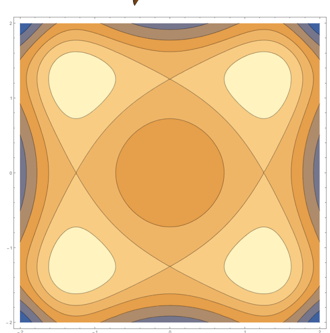
at the point $(1, 1, 1)$.



b) (5 points) Find the tangent line to the curve

$$\sin(x^2) + \sin(y^2) = c$$

which passes through the point $(1, 1)$.



Problem 9) (10 points)

The following two integrals are called "Mad Max" integrals because they were written while watching that movie:

a) (5 points) Integrate

$$\int_0^1 \int_{\arcsin(y)}^{\pi/2} \frac{xy}{\sin(x)} \, dxdy \; .$$

b) (5 points) Integrate the double integral

$$\int\int_R \sin(x^2+y^2) \, dxdy$$

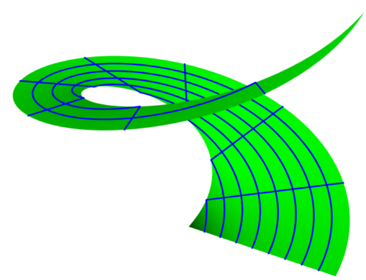
where R is the disk of radius $\sqrt{\pi/2}$.

Problem 10) (10 points)

We compute the **surface area** of the surface

$$\vec{r}(u,v) = [v\cos(u), v\sin(u), u]$$

over the region $R : 0 \leq u \leq 2\pi, u \leq v \leq 2\pi$.



a) (5 points) First verify that the integral is of the form

$$\iint_R \sqrt{1+v^2} \, dudv \; .$$

b) (5 points) Now compute the surface area integral.