MULTIVARIABLE CALCULUS

MATH S-21A

Unit 3: Cross product

Lecture

3.1. The **cross product** of two vectors $\vec{v} = [v_1, v_2]$ and $\vec{w} = [w_1, w_2]$ in the plane $\mathbb{R}^2$ is the **scalar** $\vec{v} \times \vec{w} = v_1 w_2 - v_2 w_1$. One can remember this as the determinant of a 2×2 matrix $A = \begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix}$, the product of the diagonal entries minus the product of the side diagonal entries.

3.2.

Definition: The **cross product** of two vectors $\vec{v} = [v_1, v_2, v_3]$ and $\vec{w} = [w_1, w_2, w_3]$ in space is defined as the **vector**

$$\vec{v} \times \vec{w} = [v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1] \,.$$

We can write the product also as a "determinant":

$$\begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \begin{bmatrix} i & & \\ & v_2 & v_3 \\ & w_2 & w_3 \end{bmatrix} - \begin{bmatrix} & j & \\ v_1 & & v_3 \\ w_1 & & w_3 \end{bmatrix} + \begin{bmatrix} & & k \\ v_1 & v_2 & \\ w_1 & w_2 & \end{bmatrix}$$

which is $\vec{i}(v_2 w_3 - v_3 w_2) - \vec{j}(v_1 w_3 - v_3 w_1) + \vec{k}(v_1 w_2 - v_2 w_1)$ using the notation $\vec{i} = [1, 0, 0]$, $\vec{j} = [0, 1, 0]$ and $\vec{k} = [0, 0, 1]$.

3.3. Examples: the cross product of $[1, 2]$ and $[4, 5]$ is $[5 - 8] = [-3]$. The cross product of $[1, 2, 3]$ and $[4, 5, 1]$ is $[-13, 11, -3]$.

3.4. Unlike the dot product which is commutative, the cross product is **anti-commutative:** $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

Theorem: In $\mathbb{R}^3$, the vector $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$ and has length $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$.

Proof. Check orthogonality using the dot product $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$. The length formula follows from the **Lagrange identity** $|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2 |\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2$ which is also called **Cauchy-Binet** formula. This can be done by direct computation in class. To finish up, use $|\vec{v} \cdot \vec{w}| = |\vec{v}||\vec{w}|| \cos(\alpha)|$. ∎

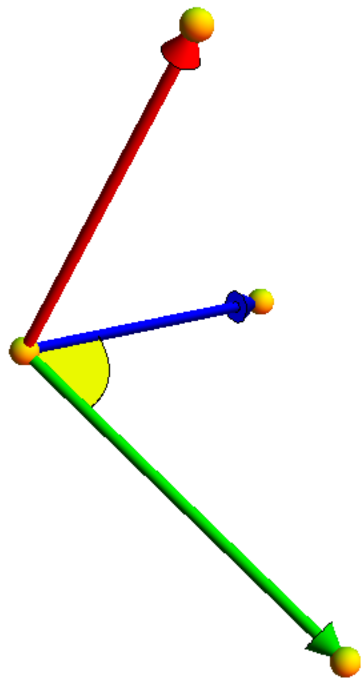

FIGURE 1. The cross product produces a vector perpendicular to two vectors. The length of the vector is the area of the parallelogram.

3.5. The statement can intuitively also to be seen by choosing a coordinate system in which the vectors are given as In that special case $\vec{v} = [a, 0, 0]$ and $\vec{w} = [b\cos(\alpha), b\sin(\alpha), 0]$, we have $\vec{v} \times \vec{w} = [0, 0, ab\sin(\alpha)]$ which has length $|ab\sin(\alpha)|$. This argument however assumes that the cross product does not change, if we change the coordinate system.

3.6. The absolute value respectively length $|\vec{v} \times \vec{w}|$ defines the **area of the parallelogram** spanned by $\vec{v}$ and $\vec{w}$. As stated as a **definition**, nothing needs to be proven. The definition fits with our common intuition we have about area because $|\vec{w}|\sin(\alpha)$ is the height of the parallelogram with base length $|\vec{v}|$.

3.7. The **trigonometric sin-formula** relates the side lengths a, b, c and angles α, β, γ of a general triangle:

Theorem: $\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$.

Proof. We can express the doubled area of the triangle in three different ways:

$$ab\sin(\gamma) = bc\sin(\alpha) = ac\sin(\beta) .$$

Divide the first equation by $\sin(\gamma)\sin(\alpha)$ to get one identity. Divide the second equation by $\sin(\alpha)\sin(\beta)$ to get the second identity. ☐

3.8. It follows from the sin-formula and the fact that $\sin(\alpha) = 0$ if $\alpha = 0$ or $\alpha = \pi$ that $\vec{v} \times \vec{w}$ is zero if and only if $\vec{v}$ and $\vec{w}$ are **parallel**, that is if $\vec{v} = \lambda \vec{w}$ for some real λ. The cross product can be used to check whether two vectors are parallel or not. Note that v and $-v$ are considered parallel even so sometimes the notion **anti-parallel** is used.

3.9.

Definition: The scalar $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is called the **triple scalar product** of $\vec{u}, \vec{v}, \vec{w}$. The absolute value of $[\vec{u}, \vec{v}, \vec{w}]$ defines the **volume of the parallelepiped** spanned by $\vec{u}, \vec{v}, \vec{w}$. The **orientation** of three vectors is defined as the sign of $[\vec{u}, \vec{v}, \vec{w}]$. It is positive if the three vectors define a **right-handed** coordinate system. It is zero if the vectors are in one plane.

3.10. We have **defined** volume and orientation using dot and cross product. Why does this fit with our intuition? The value $h = |\vec{u} \cdot \vec{n}|/|\vec{n}|$ is the height of the parallelepiped if $\vec{n} = (\vec{v} \times \vec{w})$ is a normal vector to the ground parallelogram of area $A = |\vec{n}| = |\vec{v} \times \vec{w}|$. The volume of the parallelepiped is $hA = (\vec{u} \cdot \vec{n}/|\vec{n}|)|\vec{v} \times \vec{w}|$ which simplifies to $\vec{u} \cdot \vec{n} = |(\vec{u} \cdot (\vec{v} \times \vec{w})|$ which is the absolute value of the triple scalar product. The vectors $\vec{v}, \vec{w}$ and $\vec{v} \times \vec{w}$ form a **right handed coordinate system**. If the first vector $\vec{v}$ is your thumb, the second vector $\vec{w}$ is the pointing finger then $\vec{v} \times \vec{w}$ is the third middle finger of the right hand. For example, the vectors $\vec{i}, \vec{j}, \vec{i} \times \vec{j} = \vec{k}$ form a right handed coordinate system. Since the triple scalar product is linear with respect to each vector, we also see that volume is additive. Adding two equal parallelepipeds together for example gives a parallelepiped with twice the volume.

Examples

3.11. Problem: Find the volume of the parallelepiped which has the vertices $O = (1, 1, 0), P = (2, 3, 1), Q = (4, 3, 1), R = (1, 4, 1)$. **Answer**: the solid is spanned by the vectors $\vec{u} = [1, 2, 1]$, $\vec{v} = [3, 2, 1]$, and $\vec{w} = [0, 3, 1]$. We get $\vec{v} \times \vec{w} = [-1, -3, 9]$ and $\vec{u} \cdot (\vec{v} \times \vec{w}) = 2$. The volume is 2.

3.12. Problem: Two apples have the same shape, but one has a 3 times larger diameter. What is their weight ratio? **Answer**. For a cuboid spanned by $[a, 0, 0]$ $[0, b, 0]$ and $[0, 0, c]$, the volume is the triple scalar product abc. If a, b, c are all tripled, the volume gets multiplied by a factor 27. Now cut each apple into parallelepipeds, the larger one with slices 3 times as large too. Since each of the larger pieces has 27 times the volume of the smaller, also the apple is 27 times heavier!

3.13. Problem. A **3D scanner** is used to build a 3D model of a face. It detects a triangle which has its vertices at $P = (0, 1, 1), Q = (1, 1, 0)$ and $R = (1, 2, 3)$. Find the area of the triangle. **Solution.** We have to find the length of the cross product of $\vec{PQ}$ and $\vec{PR}$ which is $[1, -3, 1]$. The length is $\sqrt{11}$. The triangle has half the area $\sqrt{11}/2$.

3.14. Problem. The scanner now detects an other point $A = (1, 1, 1)$. On which side of the triangle is it located if the cross product of $\vec{PQ}$ and $\vec{PR}$ is considered the up-direction. **Solution.** The cross product is $\vec{n} = [1, -3, 1]$. We have to see whether the vector $\vec{PA} = [1, 0, 0]$ points into the direction of $\vec{n}$ or not. To see that, we have to form the dot product. It is 1 so that indeed, A is "above" the triangle. Note that a triangle in space a priori does not have an orientation. We have to tell, what direction is "up". That is the reason that file formats for 3D printing like contain the data for three points in space as well as a vector, telling the direction.

Homework

This homework is due on Tuesday, 6/29/2021.

Problem 3.2: A three dimensional analogue of a right angle triangle is a tetrahedral shape P, Q, R, O where all angles at O are right angles. Lets assume $O = (0, 0, 0)$ and $P = (3, 0, 0), Q = (0, 5, 0), R = (0, 0, 7)$. The **3D Pythagoras theorem** stats that the sum of three triangle area triangle squared is the area of the square of the triangle PQR. That is $|OPQ|^2 + |OQR|^2 + |ORP|^2 = |PQR|^2$. Verify it in the example, then verify it in general for $P = (a, 0, 0), Q = (0, b, 0), R = (0, 0, c)$.

Problem 3.2: a) Find a unit vector perpendicular to the space diagonal $[1, 1, 1]$ and the face diagonal $[1, 0, 1]$ of the cube.
b) Find the volume of the parallelepiped for which the base parallelogram is given by the points $P = (5, 2, 2), Q = (3, 1, 2), R = (1, 4, 2), S = (-1, 3, 2)$ and which has an edge connecting P with $T = (5, 6, 8)$.
c) Find the area of the base and use b) to get the height of the parallelepiped.

Problem 3.3: To find the equation $ax + by + cz = d$ for the plane which contains the point $P = (1, 2, 3)$ as well as the line which passes through $Q = (3, 4, 4)$ and $R = (1, 1, 2)$, we find a vector $[a, b, c]$ normal to the plane and fix d so that P is in the plane.

Problem 3.4: Verify the "BAC minus CAB" formula (due to Lagrange) $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$ for general vectors $\vec{a}, \vec{b}, \vec{c}$ in space.

Problem 3.5: A product $*$ is said to satisfy the **cancellation property** if for all x, y and all $z \neq 0$ the relation $x * z = y * z$ holds, them $x = y$.
a) Does the dot product satisfy the cancellation property?
b) Does the cross product satisfy the cancellation property?