MULTIVARIABLE CALCULUS

MATH S-21A

Unit 6: Parametrized Surfaces

Lecture

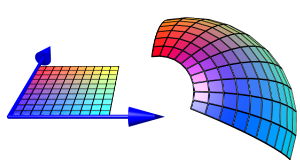
6.1. Surfaces can be described in two fundamentally different ways: first as **level surfaces** $g(x, y, z) = c$ and then through **parametrization**. What we have seen already for planes can be done more generally for other surfaces. Let us first look at the general setup:

Definition: A **parametrization** of a surface is a vector-valued function

$$\vec{r}(u, v) = [x(u, v), y(u, v), z(u, v)] ,$$

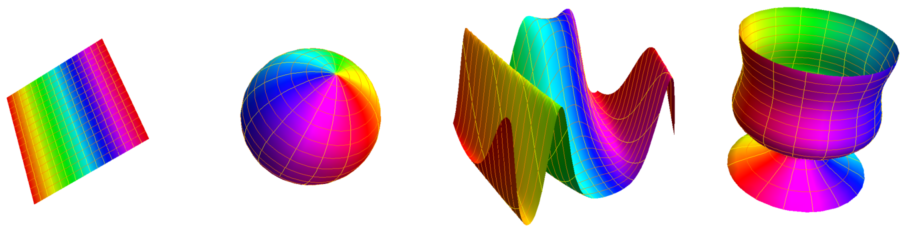
where $x(u, v), y(u, v), z(u, v)$ are three functions of two variables. The **parameters** u, v serve as coordinates on the surface. If we plug in concrete values like $u = 3, v = 2$ for example in a function $\vec{r}(u, v) = [u - 2, v^2, u^3 - v]$, we get a concrete point $\vec{r}(3, 2) = [1, 4, 25]$ in $\mathbb{R}^3$.

6.2. Because two parameters u and v are involved, the map $\vec{r}$ is also called uv**-map**. And like uv-light, it looks cool. If you like a fancy description, a parametrization is a map from $\mathbb{R}^2$ to $\mathbb{R}^3$. A **parametrized surface** is the image of the uv-map. The domain R of the uv-map is called the **parameter domain**. The parametrization is what you are **doing**, the surface itself is something you **see**. There are many different parametrizations of the same surface.



Definition: If the first parameter u is kept constant, then $v \mapsto \vec{r}(u, v)$ is a curve on the surface. Similarly, if v is constant, then $u \mapsto \vec{r}(u, v)$ traces a curve the surface. These curves are called **grid curves**.

Parametric surfaces can become complex. In that case, it is better to explored them with the help of a computer. The following four examples are important building blocks for more general surfaces.



Definition: A point $(x, y) \neq (0, 0)$ in the plane has the **polar coordinates** $r = \sqrt{x^2 + y^2}, \theta$, where θ is the angle from the positive x-axes to the point in counter clockwise direction. For $x > 0, y > 0$ it is $\arctan(y/x)$. In general $(x, y) = (r\cos(\theta), r\sin(\theta))$. A common choice is to take $\theta \in [0, 2\pi)$. The point $((0, -1)$ has then the polar coordinates $(r, \theta) = (1, 3\pi/2)$.

6.3. Note that the formula $\theta = \arctan(y/x)$ defines the angle θ only up to an addition of an integer multiple of π. The points $(1, 2)$ and $(-1, -2)$ for example have the same θ value. In order to get the correct θ value one can could take $\arctan(y/x)$ in $(-\pi/2, \pi/2]$, where $\pi/2$ is the limit when $x \to 0^+$, then add π if $x < 0$ or if $x = 0$ and $y < 0$.

Definition: The coordinate system obtained by representing points in space as

$$(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$$

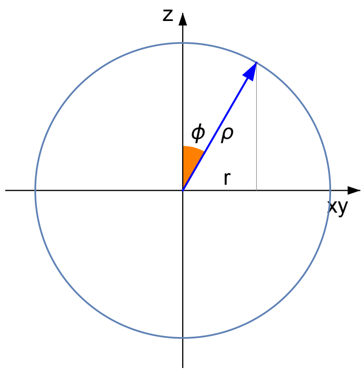
is called the **cylindrical coordinate system**.

Definition: **Spherical coordinates** use the distance ρ to the origin as well as two angles θ and ϕ called **Euler angles**. The first angle θ is the angle we have used in polar coordinates. The second angle, ϕ, is the angle between the vector $\vec{OP}$ and the z-axis. A point has the **spherical coordinate**

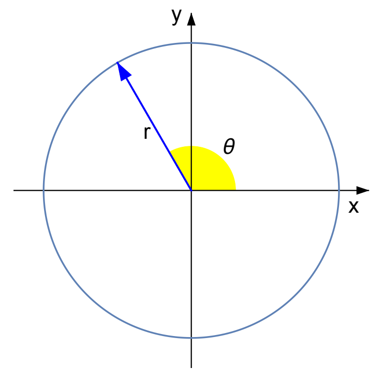
$$(x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)) .$$

We always use $0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi, \rho \geq 0$.

The following figures allow you to derive the formulas. The distance to the z axes is $r = \rho\sin(\phi)$ and the height $z = \rho\cos(\phi)$ can be read off by the left picture, the coordinates $x = r\cos(\theta), y = r\sin(\theta)$ can be seen in the right picture.



$$
\begin{aligned}
x &= \rho\cos(\theta)\sin(\phi),\\
y &= \rho\sin(\theta)\sin(\phi),\\
z &= \rho\cos(\phi)
\end{aligned}
$$



Examples

6.4. A plane has the parametrization $\vec{r}(s,t) = \vec{OP} + s\vec{v} + t\vec{w}$ and the implicit equation $ax + by + cz = d$. To get from parametric to implicit, find the normal vector $\vec{n} = \vec{v} \times \vec{w}$. To get from implicit to parametric, find two vectors $\vec{v}, \vec{w}$ normal to the vector $\vec{n}$. For example, find three points P, Q, R on the surface and form $\vec{u} = \vec{PQ}, \vec{v} = \vec{PR}$.

6.5. The **sphere** $\vec{r}(u,v) = [a,b,c] + [\rho\cos(u)\sin(v), \rho\sin(u)\sin(v), \rho\cos(v)]$ can be brought into the implicit form by finding the center and radius $(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$.

6.6. The parametrization of a graph is $\vec{r}(u,v) = [u, v, f(u,v)]$. It can be written in implicit form as $z - f(x,y) = 0$.

6.7. The surface of revolution is in parametric form given as $\vec{r}(u,v) = [g(v)\cos(u), g(v)\sin(u), v]$. It has the implicit description $\sqrt{x^2+y^2} = r = g(z)$ which can be rewritten as $x^2 + y^2 = g(z)^2$.

6.8. Here are some level surfaces in cylindrical coordinates:

> $r = 1$ is a **cylinder**, $r = |z|$ is a **double cone**, $r^2 = z$ **elliptic paraboloid**, $\theta = 0$ is a **half plane**, $r = \theta$ is a **rolled sheet of paper**.

> $r = 2 + \sin(z)$ is an example of a **surface of revolution**.

6.9. Here are some level surfaces described in spherical coordinates:

> $\rho = 1$ is a **sphere**, the surface $\phi = \pi/4$ is a **single cone**, $\rho = \phi$ is an **apple shaped surface** and $\rho = 2 + \cos(3\theta)\sin(\phi)$ is an example of a **bumpy sphere**.

Homework

This homework is due on Tuesday, 7/8/2021.

Problem 6.1: Find a parametrization $\vec{r}(t,s)$ for the plane which contains the three points $P = (-9,7,1)$,$Q = (2,2,1)$ and $R = (1,3,5)$.

Problem 6.2: Plot the surface with the parametrization

$$\vec{r}(u,v) = [(3 + v\cos(u/2))\cos(u), (3 + v\cos(u/2))\sin(u), -v\sin(u/2)]$$

with $-2 \leq v \leq 2$ and $0 \leq u \leq 2\pi$. You can use technology if you like. Do you recognize the shape of the surface? It is famous and also has a cameo in a Marvel movie.

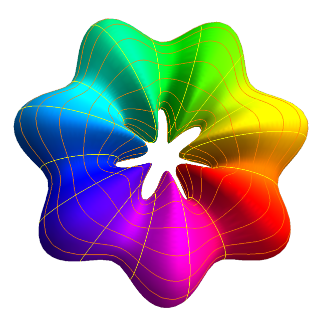
Problem 6.3: a) Find a parametrisations of the lower half of the ellipsoid $x^2/36+y^2/16+(z-4)^2 = 1$ by using that the surface is a graph $z = f(x,y)$ on a suitable domain.
b) Find a second parametrization but use angles ϕ, θ similarly as for the sphere.

Problem 6.4: Find a parametrisation of the **torus candy** (www.math-candy.com) given as the set of points which have distance $6 + 2\cos(7\theta)$ from the circle

$$[10\cos(\theta), 10\sin(\theta), 0] ,$$

where θ is the angle occurring in cylindrical and spherical coordinates. We can assure you that the candy melts wonderfully on your tongue.

Hint: Use r, the distance of a point $(x,y,)$ to the z-axis. This distance is $r = (10 + (6 + 2\cos(7\theta))\cos(\psi))$ if ψ is the angle for the circle winding around the candy. You can also use that $z = (6 + 2\cos(7\theta))\sin(\psi)$. To finish the parametrization problem, translate back to Cartesian coordinates.



Problem 6.5: a) What is the equation for the surface $x^2+y^2 = z^2+y/x$ in cylindrical coordinates?
b) Describe in words or draw a sketch of the surface whose equation is $\rho = |\sin(4\phi)|$ in spherical coordinates (ρ,θ,ϕ).

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