MULTIVARIABLE CALCULUS

MATH S-21A

Unit 21: Green's theorem

Lecture

21.1. Green's theorem is the second integral theorem in two dimensions. When we do multi-variable calculus in two dimensions, there are only **two** derivatives and **two** integral theorems: the **fundamental theorem of line integrals** as well as **Green's theorem**. You might be used to think about the two-dimensional case as a special case of the xy-plane in three space, but we insist on remaining two dimensional. [1]

21.2. Remember that the **curl** of a vector field $\vec{F}(x,y) = [P(x,y), Q(x,y)]$ is the scalar field $\text{curl}(\vec{F})(x,y) = \nabla \times \vec{F} = Q_x(x,y) - P_y(x,y)$. It measures the **vorticity** of the vector field at (x,y). For example, for $\vec{F}(x,y) = [x^3 + y^2, y^3 + x^2 y]$, we have $\text{curl}(F)(x,y) = 2xy - 2y$.

> **Theorem: Green's theorem**: If $\vec{F}(x,y) = [P(x,y), Q(x,y)]$ is a vector field and G is a region for which the boundary C is a curve parametrized so that G is "to the left", then
> $$\int_C \vec{F} \cdot \vec{dr} = \int\int_G \text{curl}(\vec{F})\ dxdy\ .$$

21.3. Take a square $G = [x, x+h] \times [y, y+h]$ with small $h > 0$. The line integral of $\vec{F} = [P, Q]$ along the boundary is $\int_0^h P(x+t,y)dt + \int_0^h Q(x+h, y+t)\ dt - \int_0^h P(x+t, y+h)\ dt - \int_0^h Q(x, y+t)\ dt$. It measures the "circulation" at the place (x,y). Because $Q(x+h,y) - Q(x,y) \sim Q_x(x,y)h$ and $P(x,y+h) - P(x,y) \sim P_y(x,y)h$, the line integral is $(Q_x - P_y)h^2$ is $\int_0^h \int_0^h \text{curl}(\vec{F})\ dxdy$ with an error of the order h^3 or smaller. Now take a region G with area $|G|$ and chop it into small squares of size h. We need about $|G|/h^2$ such squares. Summing up all the line integrals around the boundaries is the sum of the line integral along the boundary of G because of the cancellations in the interior. On the boundary, it is a Riemann sum of the line integral along the boundary. The sum of the curls of the squares is a Riemann sum approximation of the double integral $\int\int_G \text{curl}(\vec{F})\ dxdy$. Taking the limit $h \to 0$ gives Greens theorem.

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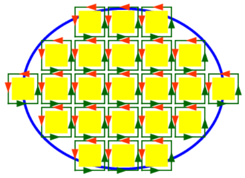
[1] It is better to think about two dimensions as if we were flat-landers unaware about the third dimension. If we speak about "the plane", this is our universe, we are ignorant about 3 space. Edwin Abbot's Flatland is a 1884 romance plays in two dimensions.



21.4. **George Green** lived from 1793 to 1841. Unfortunately, we don't have a single picture of him. He was a physicist, a self-taught mathematician as well as a miller. His work greatly contributed to modern physics.

21.5. A special case is if $\vec{F}$ is a gradient field $\vec{F} = \nabla f$. Then, both sides of Green's theorem are zero: $\int_C \vec{F} \cdot \vec{dr}$ is zero by the fundamental theorem for line integrals. And $\int\int_G \mathrm{curl}(\vec{F}) \cdot dA$ is zero because $\mathrm{curl}(\vec{F}) = \mathrm{curl}(\mathrm{grad}(f)) = 0$.

21.6. If $\vec{F}(x,y) = \nabla f$ is a gradient field then the curl is zero because if $P(x,y) = f_x(x,y), Q(x,y) = f_y(x,y)$ and $\mathrm{curl}(\vec{F}) = Q_x - P_y = f_{yx} - f_{xy} = 0$ by Clairaut. $\vec{F}(x,y) = [x+y, yx]$ for example is not a gradient field because $\mathrm{curl}(\vec{F}) = y - 1$.

21.7. The already established **Clairaut identity**

$\mathrm{curl}(\mathrm{grad}(f)) = 0$

21.8. This can also be remembered by writing $\mathrm{curl}(\vec{F}) = \nabla \times \vec{F}$ and $\mathrm{curl}(\nabla f) = \nabla \times \nabla f$. Use now that cross product of two identical vectors is 0. Working with ∇ as a vector is called **nabla calculus** which can serve as a mnemonic.

21.9. It had been a consequence of the fundamental theorem of line integrals that:

If $\vec{F}$ is a gradient field then $\mathrm{curl}(\vec{F}) = 0$ everywhere.

21.10. Is the converse true? Here is the answer:

Definition: A region R is called **simply connected** if every closed loop in R can be pulled together continuously within R to a point inside R.

21.11. $R = \{x^2 + y^2 \leq 1\}$ is simply connected, $O = \{3 \leq x^2 + y^2 \leq 4\}$ is not.

If $\mathrm{curl}(\vec{F}) = 0$ in a simply connected region G, then $\vec{F}$ is a gradient field.

Proof. Given a closed curve C in G enclosing a region R. Green's theorem assures that for any gradient field $\vec{F}$ we have $\int_C \vec{F}\, \vec{dr} = 0$. So $\vec{F}$ has the closed loop property in G. This is equivalent to the fact that line integrals are path independent. In that case $\vec{F}$ is therefore a gradient field: one can get $f(x, y)$ by taking the line integral from an arbitrary point O to (x, y). In the homework, you look at an example of a not simply connected region where the $\text{curl}(\vec{F}) = 0$ does not imply that $\vec{F}$ is a gradient field.

Examples

21.12. Problem: Find the line integral of $\vec{F}(x, y) = [x^2 - y^2, 2xy] = [P, Q]$ along the boundary of the rectangle $[0, 2] \times [0, 1]$. **Solution:** $\text{curl}(\vec{F}) = Q_x - P_y = 2y + 2y = 4y$ so that $\int_C \vec{F}\, \vec{dr} = \int_0^2 \int_0^1 4y\, dydx = 2y^2|_0^1 x|_0^2 = 4$.

21.13. Problem: Find the area of the region enclosed by

$$\vec{r}(t) = [\frac{\sin^2(\pi t)}{t}, t^2 - 1]$$

for $-1 \leq t \leq 1$. To do so, use Greens theorem with the vector field $\vec{F} = [0, x]$.

21.14. Green's theorem allows to express the coordinates of the **centroid** = center of mass

$$(\int \int_G x\, dA/A, \int \int_G y\, dA/A)$$

using line integrals. With $\vec{F} = [0, x^2/2]$ we have $\int \int_G x\, dA = \int_C \vec{F}\, \vec{dr}$.

21.15. An important application of Green is **area computation**: Take a vector field like $\vec{F}(x, y) = [P, Q] = [0, x]$ which has constant vorticity $\text{curl}(\vec{F})(x, y) = 1$. For $\vec{F}(x, y) = [0, x]$, the right hand side in Green's theorem is the **area** $\text{Area}(G) = \int_C \vec{F}(\vec{r}(t))\, \vec{r}'(t)\, dt$.

21.16. Let G be the region below the graph of a function $f(x)$ on $[a, b]$. The line integral around the boundary of G is 0 from $(a, 0)$ to $(b, 0)$ because $\vec{F}(x, y) = [0, 0]$ there. The line integral is also zero from $(b, 0)$ to $(b, f(b))$ and $(a, f(a))$ to $(a, 0)$ because $P = 0$. The line integral along the curve $(t, f(t))$ is $-\int_a^b [-y(t), 0] \cdot [1, f'(t)]\, dt = \int_a^b f(t)\, dt$. Green's theorem confirms that this is the area of the region below the graph.

21.17. An engineering application is the **planimeter**, a mechanical device for measuring areas. We demonstrate it in class. Historically it had been used in medicine to measure the size of the cross-sections of tumors, in biology to measure the area of leaves or wing sizes of insects, in agriculture to measure the area of forests, in engineering to measure the size of profiles. There is a vector field $\vec{F}$ associated to the device which is obtained by placing a unit vector perpendicular to the arm). One can prove that $\vec{F}$ has vorticity 1. The planimeter calculates the line integral of $\vec{F}$ along a given curve. Green's theorem assures this is the area.

Homework

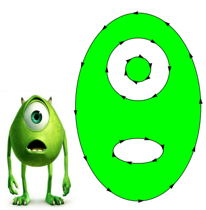
This homework is due on Tuesday, 8/3/2021.

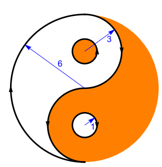
Problem 21.1: Given $f(x,y) = 144345 * x^5 + 234234 * xy^4$, compute the line integral of $\vec{F}(x,y) = [25y + 6y^2, 12xy + 10y^{445555}] + \nabla f$ along the boundary of the **Monster region** given in the picture. There are four boundary curves, oriented as shown in the picture: a large ellipse of area 16, two circles of area 1 and 2 as well as a small ellipse (the mouth) of area 3.

Problem 21.2: Find the area of the region bounded by the **hypocycloid** $\vec{r}(t) = [4\cos^3(t), 4\sin^3(t)], 0 \leq t \leq 2\pi$.

Problem 21.3: Let G be the region $x^6 + y^6 \leq 1$. Mathematica allows us to get the area as Area[ImplicitRegion[$\mathrm{x}^6 + \mathrm{y}^6 <= 1, \{\mathrm{x}, \mathrm{y}\}$]] and tells, it is $A = 3.8552$. Compute the line integral of $\vec{F}(x,y) = [x^{800} + \sin(x) - 55y, y^{12} + \cos(y) + 3x]$ counter clockwise along the boundary in terms of A.

Problem 21.4: Let C be the boundary curve of the white Yang part of the Ying-Yang symbol in the disc of radius 6. You can see in the image that the curve C has three parts, and that the orientation of each part is given. Find the line integral of the vector field $\vec{F}(x,y) = [-y + \sin(e^x), x]$ around C.





Problem 21.5: Use Green's Theorem to evaluate $\int_C [\sin(\sqrt{1+x^3}) + 21y, 121x] \cdot d\vec{r}$, where C is the boundary of the region $K(4)$. You see in the picture $K(0), K(1), K(2), K(3), K(4)$. The first $K(0)$ is an equilateral triangle of length 1. The second $K(1)$ is $K(0)$ with 3 equilateral triangles of length 1/3 added. $K(2)$ is $K(1)$ with $3 * 4^1$ equilateral triangles of length 1/9 added. **Remark.** We could now find the line integral in the limit $K = K(\infty)$, a **fractal** called the **Koch snowflake** It has dimension $\log(4)/\log(3) = 1.26\ldots$ which is between 1 and 2.