∂ "partial"

Unit 9 PDE's

(1) Partial derivation

$$f(x,y) = x^5 + yx^3$$

$$\frac{\partial}{\partial x} f(x,y) = 5x^4 + 3yx^2 = f_x$$

partial derivative w.r.t. x

$$f_y(x,y) = x^3$$

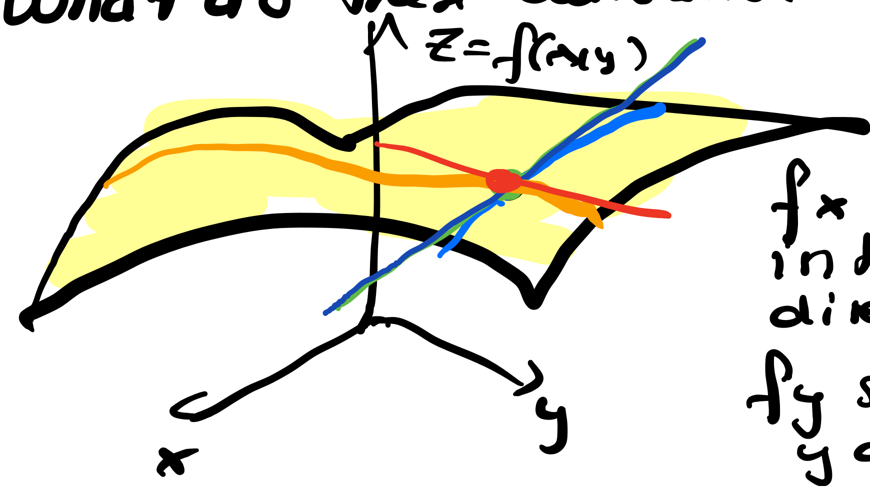
$$f_{xx}(x,y) = 20x^3 + 6xy$$

$$f_{yy}(x,y) = 0$$

$$f_{xy}(x,y) = 3x^2$$

$$f_{yx}(x,y) = 3x^2$$

What do these derivatives mean



f_x is slope in the x direction

f_y simil. in y direction

What is f_{xx}? Concavity in x direction

$z = x^2 + y^2$

$f_x = 2x$

$f_{xx} = 2$ concave u

$f_{yy} = 2$ concave up

What is the meaning of f_{xy}?

change of the x slope if y changes.

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2 Clairaut's theorem

$$\boxed{f_{xy} = f_{yx}}$$

if f_{xy}, f_{yx} are continuous functions.

Proof:

$$h f_x \approx f(x+h,y) - f(x,y)$$

$$h^2 f_{xy} = [f(x+h,y+h) - f(x,y+h)]$$
$$- [f(x+h,y) - f(x,y)]$$

$$h f_y \approx f(x,y+h) - f(x,y)$$

$$h^2 f_{yx} = [f(x+h,y+h) - f(x+h,y)]$$
$$- [f(x,y+h) - f(x,y)]$$

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on the discrete level these are the same expression. Now divide by h^2 and take the limit $h \to 0$

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upper part $\to f_{xy}$

lower part $\to f_{yx}$

If f_{xy} is not continuous there are counterexamples.

For $\frac{4x^2y^2}{x^2+y^2} = f(x,y)$

By Hospidap you can check this is continuous

$\longrightarrow$ polar coordinate.

$x = r\cos\theta$
$y = r\sin\theta$

$$\frac{4r^4\cos^2\theta\sin^2\theta}{r^2}$$

$\sin^2 2\theta \cdot r^2$

③ $\boxed{\text{PDE's}}$

You have to know
5 PDE's.

$$f_x = f \qquad \underline{\text{ODE}}$$

$$f_x = f_{xy} \qquad \underline{\text{PDE}}$$

<u>PDE</u>: equation for unknown function involving partial derivative w.r.t. at least 2 variables.

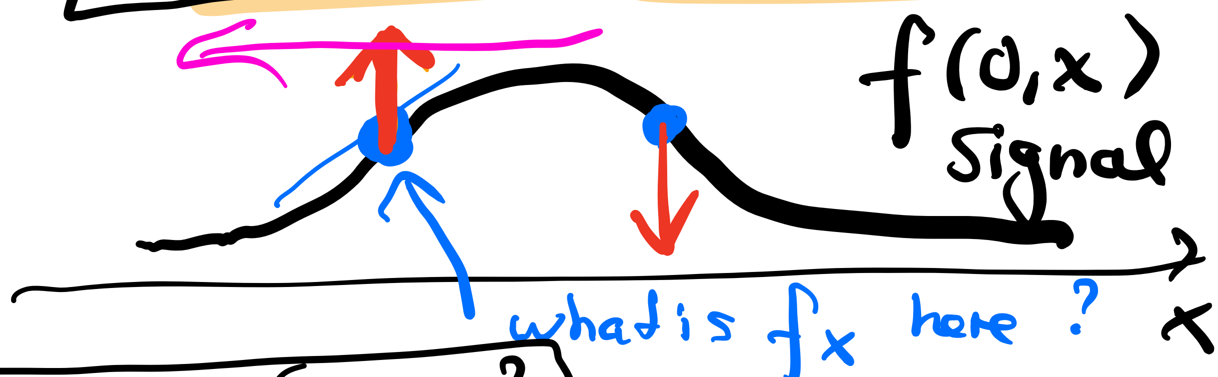
$$\boxed{f_{xy} = f_{yx}}$$

is solved by smooth function

Typically one of the variables is <u>time</u> t and the other is space x

a) **Transport equation**

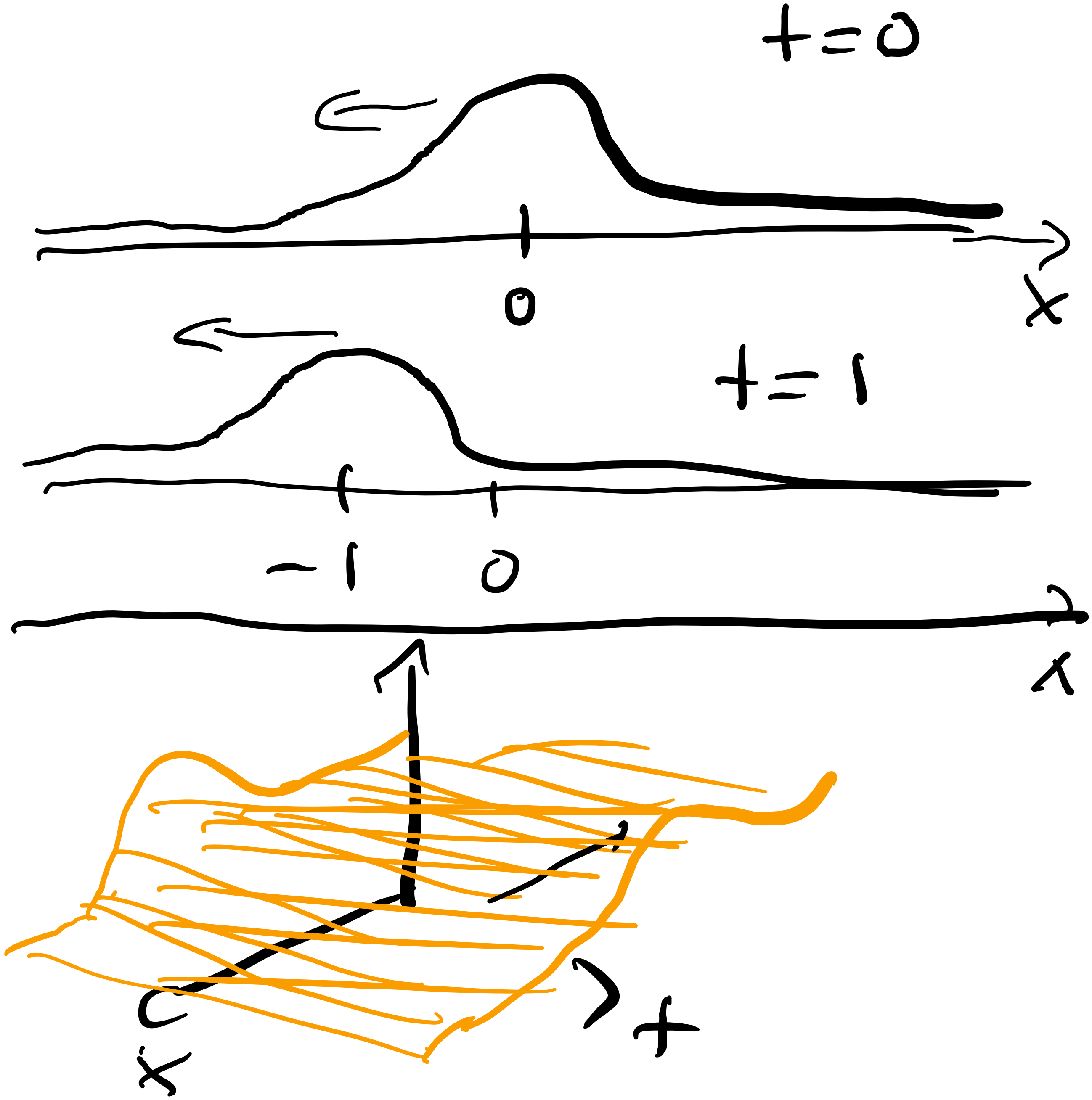
$$\boxed{f_t(t,x) = f_x(t,x)}$$



$\boxed{f(t,x) = e^{-(x+t)^2}}$ is a solution to the transport eqn.

$$f_t = -2e^{-(x+t)^2}(x+t)$$

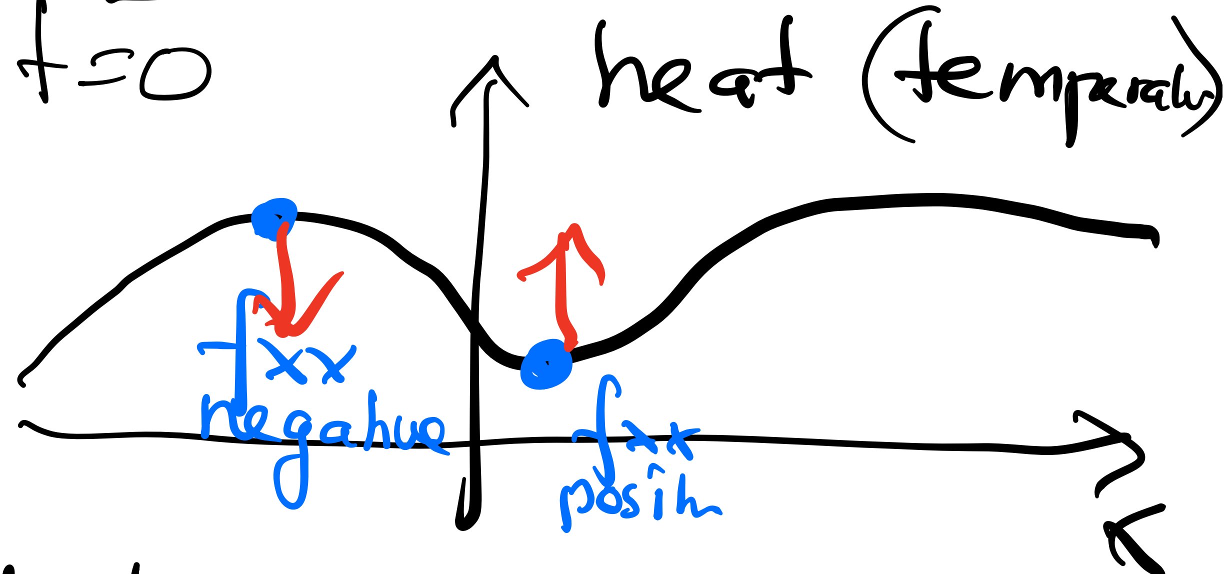
$$f_x = -2e^{-(x+t)^2}(x+t)$$



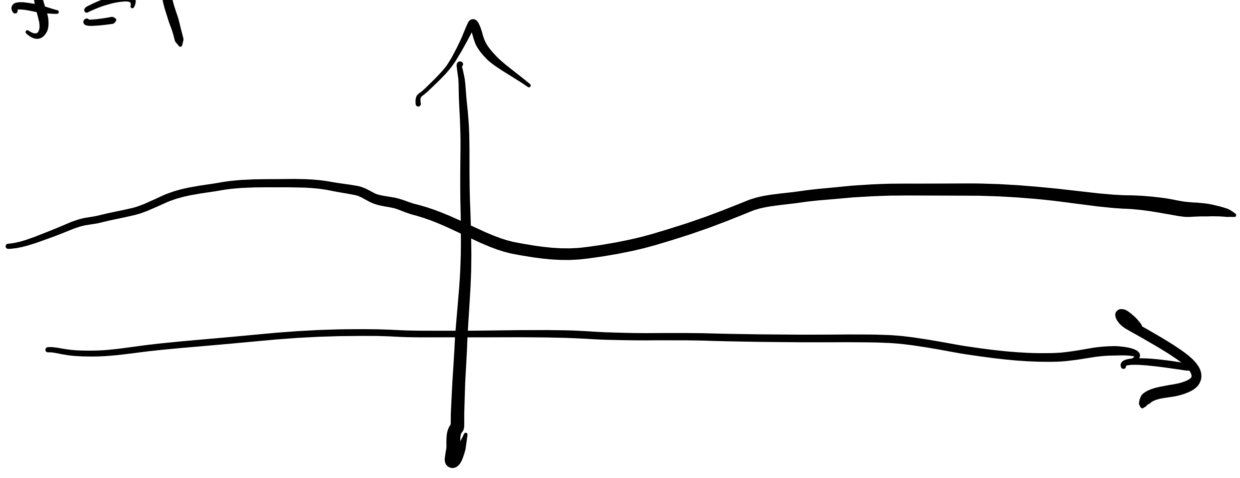
b) Heat equation

$$\boxed{f_t = f_{xx}}$$

$t=0$



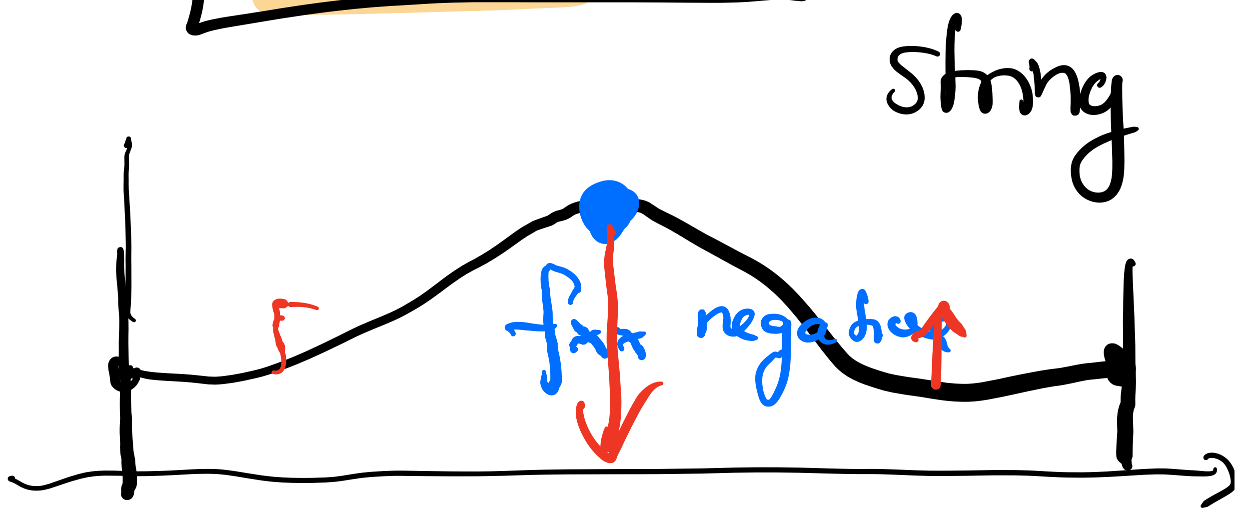
$t=1$



diffusion equation

c) Wave equation

$$\boxed{f_{tt} = f_{xx}}$$



motion of a string.

d) $$\boxed{f_{xx} + f_{yy} = 0}$$

<u>Laplace equation</u>

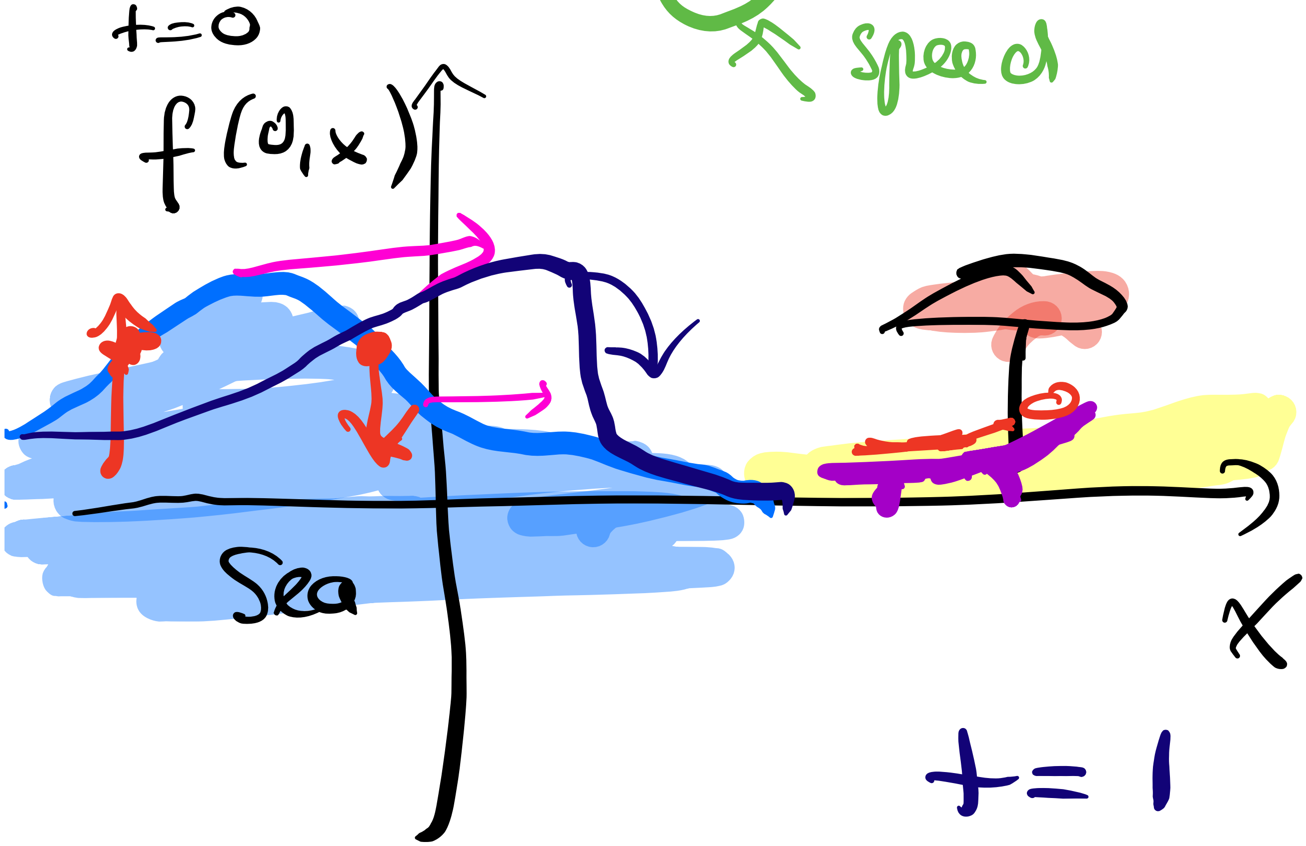
$$\Delta f = 0$$

e) $\boxed{f_t = f f_x + \nu f_{xx}}$

Burgers equation,

$$f_t = f f_x$$

speed



motion of waves.

Navier Stokes equation

$$f_t = f f_x + F$$

1'000'000 solution exists

$$i f_t = f_{xx}$$

Schrödinger equation in QM

$$f_t = f - x f_x - x^2 f_{xx}$$

Black Sholes