Unit 11 Chain rule

$$\frac{d}{dx}\sin(e^x) = \cos(e^x)e^x$$

$$\boxed{\frac{d}{dx} f(g(x)) = f'(g(x))\cdot g'(x)}$$

1 The chain rule in higher dims

f function of several variables

$\vec{r}(t)$ curve

$$\boxed{\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)}$$

E1: $f(x,y) = x^2+y^2$

$\vec{r}(t) = \begin{bmatrix} 1+\cos t \\ 1-\sin t \end{bmatrix}$, what is $\frac{d}{dt} f(\vec{r}(t))$ at $t=0$?

$$f(\vec{r}(t)) = (1+\cos(t))^2 + (1-\sin t)^2$$

$$\nabla f(\vec{r}(0)) = \nabla f(2,1) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\vec{r}\,'(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\nabla f(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

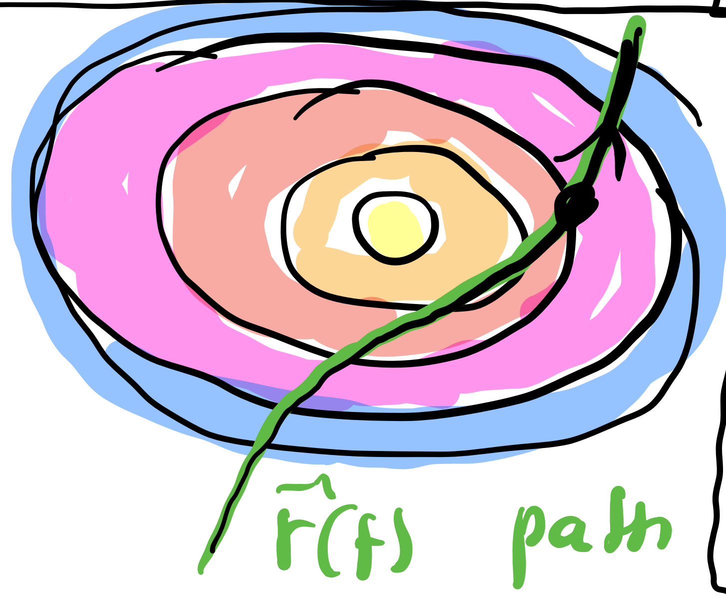
So: $\frac{d}{dt} f(\vec{r}(t))\Big|_{t=0} = -2$

$f(x,y) = x^2+y^2,$ $\quad \nabla f(x,y) = \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix}$

$\vec{r}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

$\nabla f(2,1) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

interpretation



$f(x,y)$ = height or temperature or pressure

$\frac{d}{dt} f(\vec{r}(t))$ is the rate of change of f when walking along the path $\vec{r}(t)$.

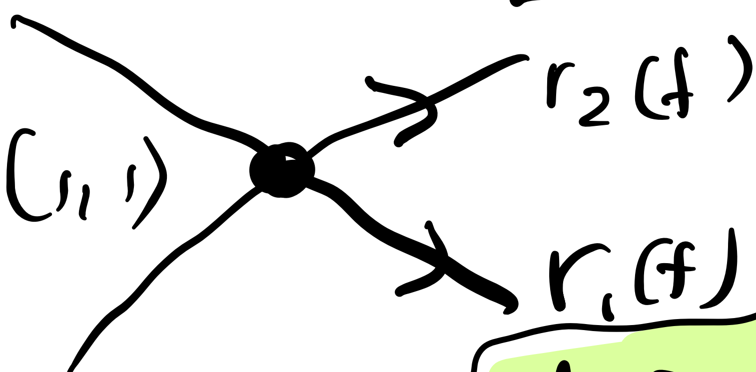
Here is a good problem!

E2: $f(x,y)$ is not known.

Two paths:

$$\vec{r}_1(t) = \begin{bmatrix} 1+t \\ 1-t \end{bmatrix}$$

$$\vec{r}_2(t) = \begin{bmatrix} 1+2t \\ 1+3t \end{bmatrix}$$



We know: $11 = \frac{d}{dt} f(\vec{r}_1(t))\Big|_{t=0}$

and $17 = \frac{d}{dt} f(\vec{r}_2(t))\Big|_{t=0}$

Find $\nabla f(1,1) = \begin{bmatrix} a \\ b \end{bmatrix}$

Solution: use the

<u>Chain rule</u>:

$$11 = \frac{d}{dt} f(\vec{r}_1(t))\Big|_{t=0} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$17 = \frac{d}{dt} f(\vec{r}_2(t))\Big|_{t=0} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\left| \begin{array}{l} 11 = a - b \\ 17 = 2a + 3b \end{array} \right|$$

<u>eliminate</u>:

$$\begin{array}{l} 33 = 3a - 3b \\ 17 = 2a + 3b \\ \hline 50 = 5a \quad a = 10 \\ b = -1 \end{array}$$

$$\nabla f = \begin{bmatrix} 10 \\ -1 \end{bmatrix}$$

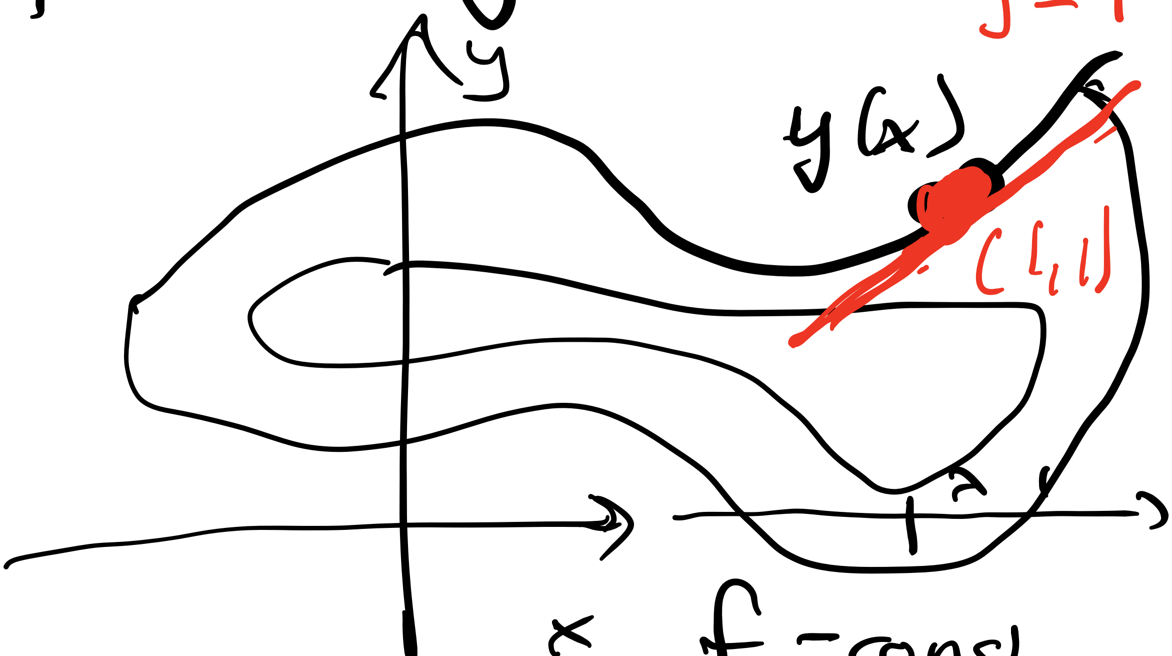
It was possible to compute the gradient even so we did not know f!

③ Implicit differentiation

Problem:

$$xy^3 + x^5y + xy^5 = 3$$

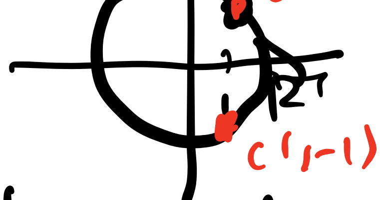
Find $y'(x)$ at $x=1$, $y=1$



Similar as in

$$x^2+y^2=2$$

Find $y'(x)$ at $x=1$, $y=1$



we could solve for y

mess

$$y = \sqrt{2-x^2}$$

$$y' = \frac{-2x}{2\sqrt{2-x^2}} = -1$$

$$\nabla f = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$$y' = -\frac{2}{2}$$

In our case, we can not solve for y.

We can still find $y'(x)$.

Take $\vec{r}(x) = \begin{bmatrix} x \\ y(x) \end{bmatrix}$ "t" $= x$

use the chain rule.

$$y^3 + x\,3y^2y' + 5x^4y + x^5y' + y^5 + x\,5y^4y' = 0$$

now solve for y'.

Easier: $\boxed{y' = -\frac{f_x}{f_y}}$

$$f(x,y) = c \qquad r(x) = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$r'(x) = \begin{bmatrix} 1 \\ y' \end{bmatrix}$$

$$\frac{d}{dx} f(x,y) = 0$$

$$= f_x \cdot 1 + f_y \cdot y' = 0$$

$$y' = \frac{-f_x}{f_y} \qquad \text{QED}$$

$$f(x,y) = xy^3 + x^5 y + xy^5$$

$$\nabla f = \begin{bmatrix} y^3 + 5x^4 y + y^5 \\ 3xy^2 + x^5 + 5xy^4 \end{bmatrix}$$

$$\nabla f(1,1) = \begin{bmatrix} 7 \\ 9 \end{bmatrix}, \quad \boxed{y' = \frac{-7}{9}}$$

4) One ring to rule them all

$$f(x,y) = xy \quad , \quad \begin{matrix} x(t) \\ y(t) \end{matrix}$$

$$\nabla f = \begin{bmatrix} y \\ x \end{bmatrix}, \quad \vec{r}\,'(t) = \begin{bmatrix} \vec{x}\,' \\ \vec{y}\,' \end{bmatrix}$$

$$\frac{d}{dt}(xy) = y x' + x y'$$

Product rule in single var

$f(x,y) = \frac{x}{y} \longrightarrow$ quotient rule

$f(x,y) = x+y \longrightarrow$ additionally

In 3D HW

$z = z(x,y)$

$f(x,y,z) = c$

$$z_x = \frac{-f_x}{f_z}$$

$$z_y = -f_y/f_z$$

implicit differentiation formulas in 3D!