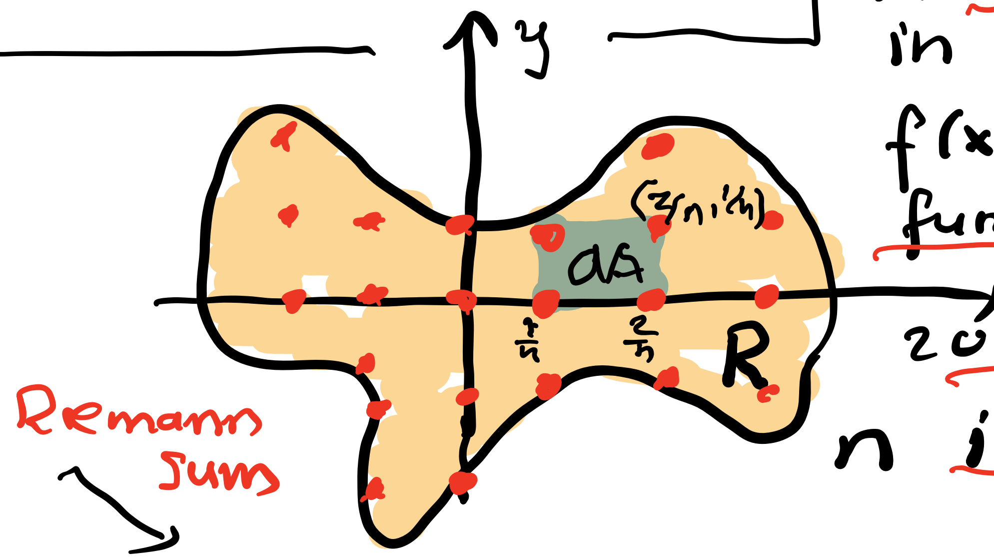
(15) **Integration in 2D**

1 Set up



R region in $\mathbb{R}^2$

$f(x,y)$ function of 2 variables

n integer

$$\sum_{(\frac{k}{n},\frac{\ell}{n}) \text{ in } R} \sum f\left(\frac{k}{n},\frac{\ell}{n}\right) \underbrace{\frac{1}{n}\frac{1}{n}}_{dA}$$

$$\Big\downarrow \, n \to \infty$$

$$\boxed{\iint_R f(x,y)\, \underbrace{dx\,dy}_{dA}}$$

Definition of Riemann integral as a limit of Riemann sums.

a) What happens, if $f(x,y)=1$?
This is the area

$$\iint_R 1 \, dx dy = \text{Area}$$

Area

compare:

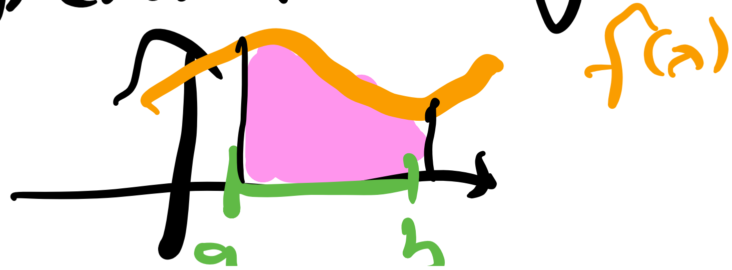
$$\int_a^b 1 \, dx = \text{Length} = b-a$$

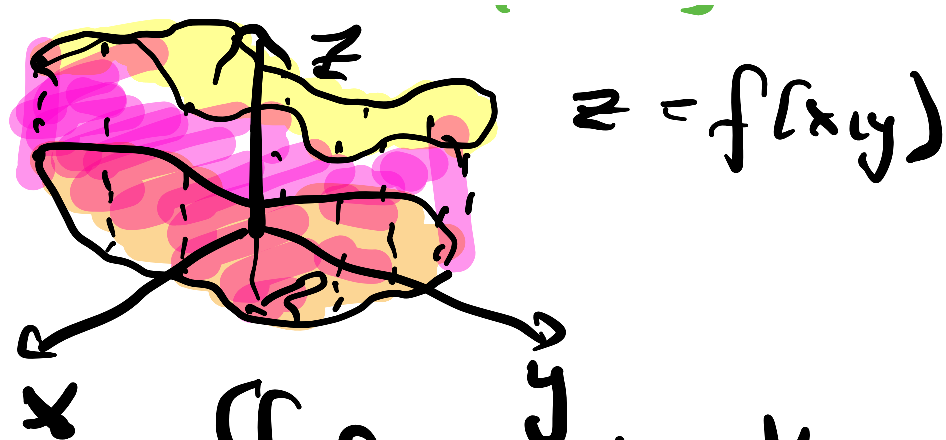
b) Is there a geometric interpretation of $\iint_R f \, dA$

analogue to

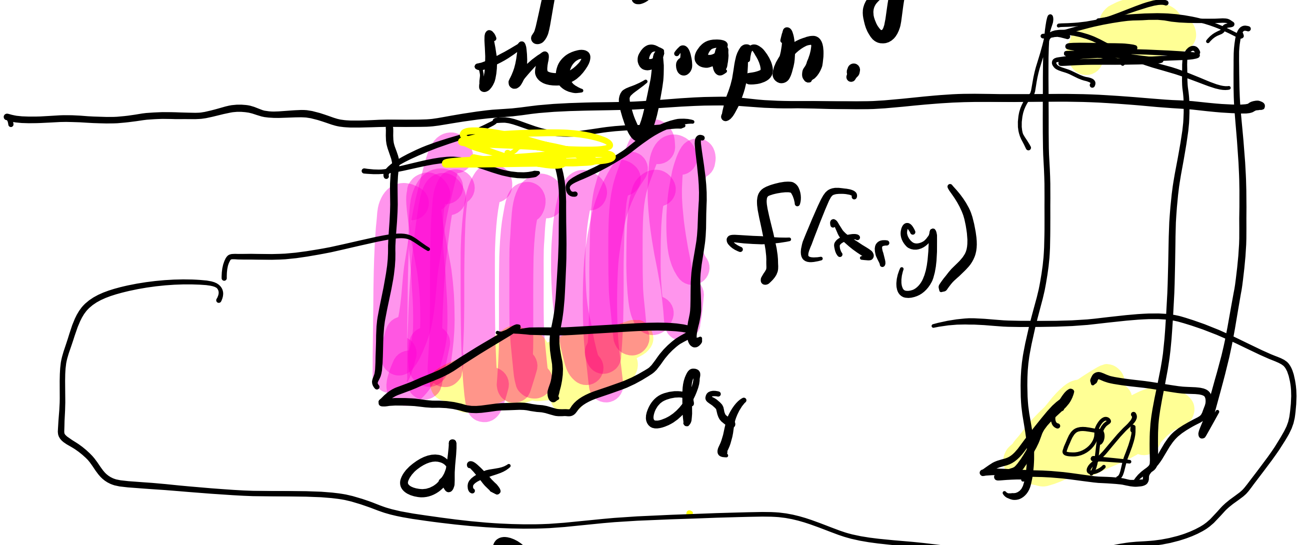
$$\boxed{\int^b f(x)\,dx}$$ had an interpretation as signed area

Volume





$\iint_R f \, dA$ is the signed volume of the region below the graph.

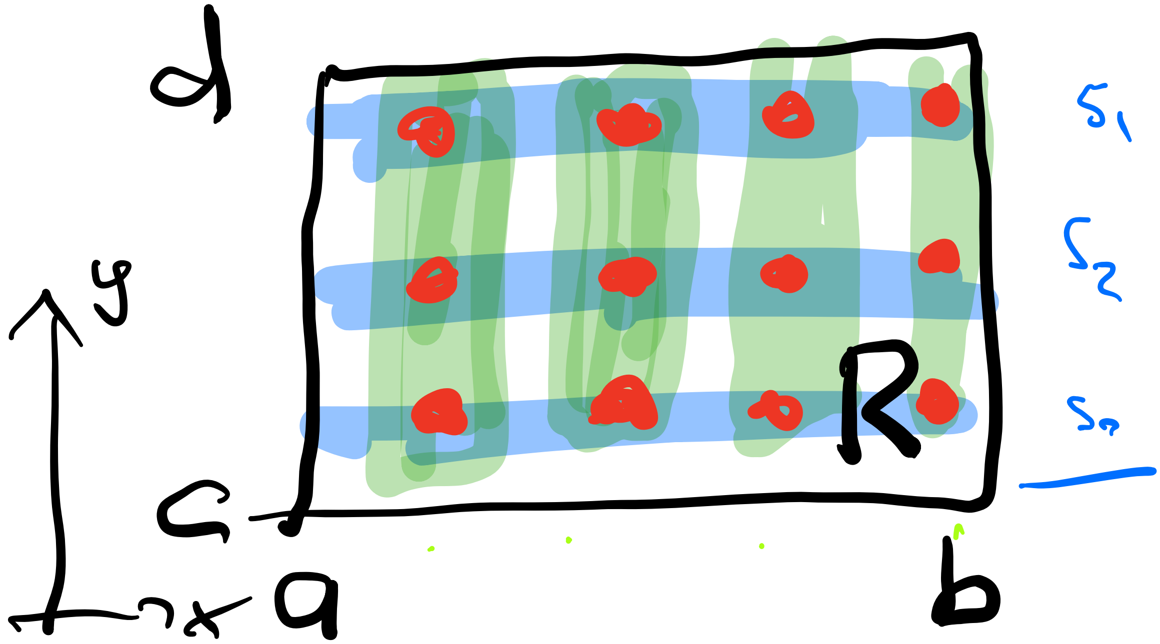


$\int f(x,y)\, dx\, dy$

is the volume of a the "skyscraper above $d\cancel{x}dy = dA$

② Fubini's theorem

Reduce double integrals to single integrals.



R is a rectangle

We can sum up the Riemann sum in two different ways.

$$\sum_{a \le \frac{k}{n} < b} \boxed{\sum_{c \le \frac{\ell}{n} \le d} f\left(\frac{k}{n}, \frac{\ell}{n}\right) \frac{1}{n}} \frac{1}{n}$$

$$\Big\downarrow n \to \infty$$

$$\sum_{a \le \frac{k}{n} < b} \Bigg(\underbrace{\int_c^d f\left(\frac{k}{n}, y\right) dy}_{g\left(\frac{k}{n}\right)} \Bigg) \frac{1}{n}$$

$$\boxed{\int_a^b \boxed{\int_c^d f(x,y)\, dy}\, dx}$$

$$\sum_{c \le \frac{\ell}{n} < d} \boxed{\sum_{a \le \frac{k}{n} < b} f\left(\frac{k}{n}, \frac{\ell}{n}\right) \frac{1}{n}} \frac{1}{n}$$

$$\Big\downarrow n \to \infty$$

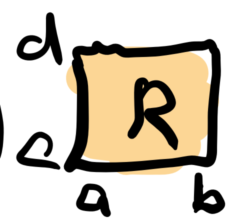
$$\sum_{c \le \frac{\ell}{n} < d} \underbrace{\int_a^b f\left(x, \frac{\ell}{n}\right) dx}_{g\left(\frac{\ell}{n}\right)} \frac{1}{n}$$

$$\int_c^d \boxed{\int_a^b f(x,y)}\, dx\, dy$$

We have proven a theorem:

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

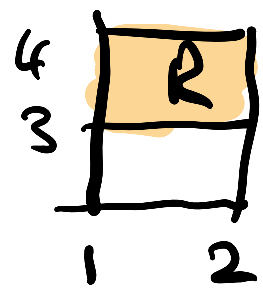
Theorem of Fubini



On rectangles, the order does not matter.

In general it will be essential

Ex: $\displaystyle\int_1^2\int_3^4 xy^2\,dy\,dx$



$$\frac{xy^3}{3}\Big|_3^4 = \frac{x(64-27)}{3}$$

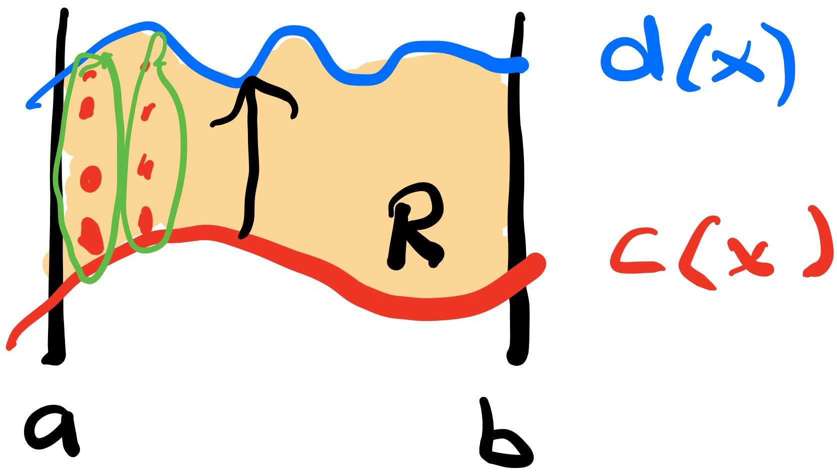
$$\frac{37}{3}\int_1^2 x\,dx = \frac{37}{3}\,\frac{x^2}{2}\Big|_1^2 = \left(\frac{4-1}{2}\right)\frac{37}{3}$$

$$\boxed{\frac{37}{2}}$$

(2) Basic regions

"type I in Stewart (old fashioned)"

a)

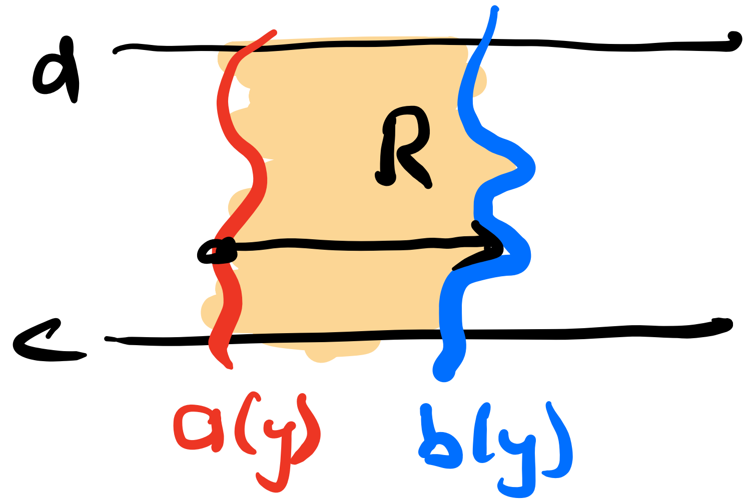


$$R = \{ (x,y) \ , \ a \le x \le b \ \text{and} \ c(x) \le y \le d(x) \}$$

<u>bottom to top region</u>

$$\int_a^b \int_{c(x)}^{d(x)} f(x,y) \, dy \, dx$$

b)



type 2 in Stewart old fashioned

$$R = \{(x,y),\ c \le y \le d,\ \text{and } a(y) \le x \le b(y)\}$$

$$\int_c^d \int_{a(y)}^{b(y)} f(x,y)\, dx\, dy$$

Left to right region

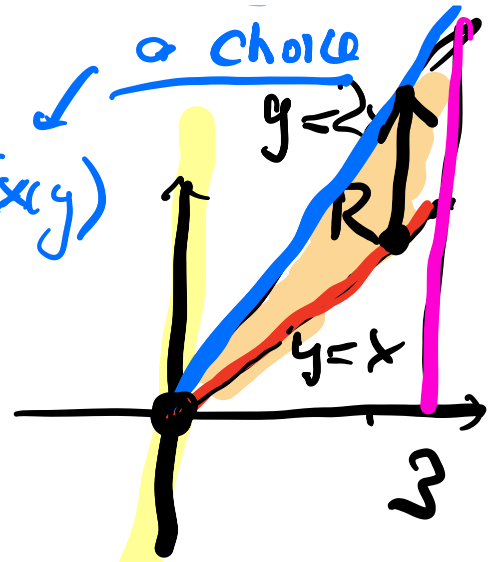
this is

④ **Examples**

a) $$\iint_R y \, dA$$

$f(x,y)$

a choice



R is bound by $\cancel{y=x}$, $y=2x$, $\boxed{x=3}$ and $x=0$

Always make a picture of the region

$$\int_0^3 \left[\int_x^{2x} y \, dy \right] dx$$

$$\frac{y^2}{2}\Big|_x^{2x} = \frac{3x^2}{2}$$

$$\int_0^3 \frac{3x^2}{2}\,dx = \frac{3x^3}{6}\Big|_0^3$$

$$= \frac{3 \cdot 27}{6}$$

$$= \boxed{\frac{27}{2}}$$

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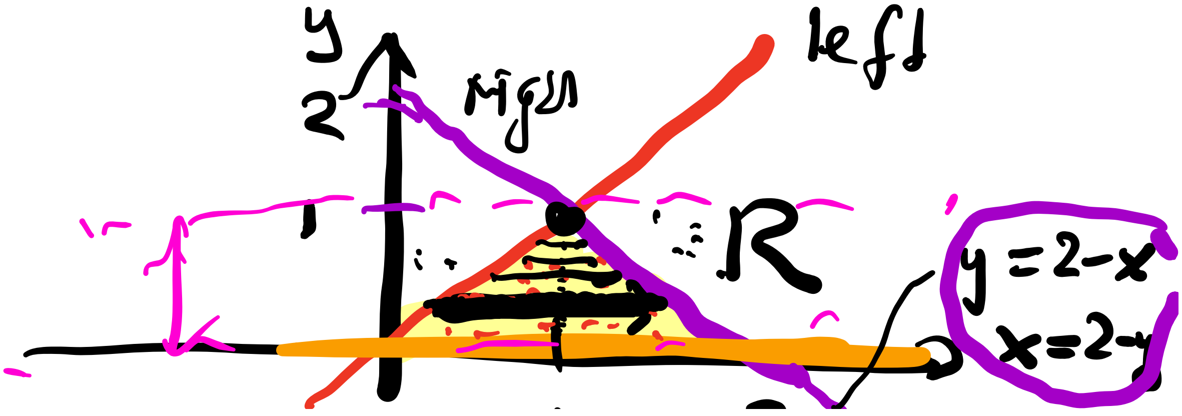
b) Find $\iint_R xy\,dA$ where R is bound by $x = y$, $y = 2 - x$ and $y = 0$



1 2

How do we set-up this integral?
Splitting things up is not elegant. We can do that in one sweep by:

Make it a <u>left to right integral.</u>

sums from left to right if y is fixed

$$\boxed{\int_0^1 \int_y^{2-y} xy \, dx} \, dy$$

$$\cancel{\int_0^1 \int_0^2 xy \, dx \, dy}$$

$$\frac{x^2 y}{2}\Big|_{y}^{2-y} = \frac{\left((2-y)^2 - y^2\right) y}{2}$$

$$\int_0^1 \frac{\left((2-y)^2 - y^2\right)}{2}\, y \, dy = \dots$$

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5) Switching the order

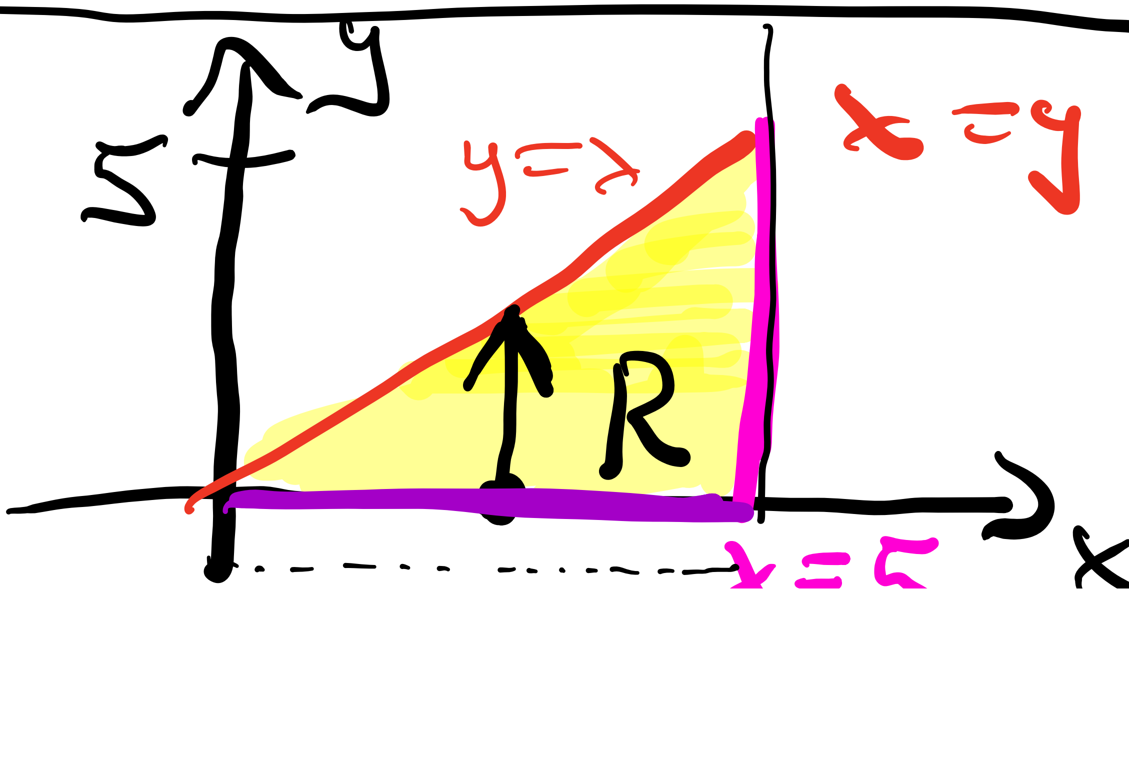
$$\int_0^5 \int_{\boxed{4}}^{\boxed{5}} e^{-x^2}\, dx \, dy$$

Find this integral!

You can't do the
inner integral

$\int e^{-x^2}dx$ is not
elementary.

next lecture: $\int_{-\infty}^{\infty} e^{-x^2}dx$



$$\int_0^5 \boxed{\int_0^x e^{-x^2}\,dy}\,dx$$

now constant

Now, we can integrate

$$\boxed{y e^{-x^2}\Big|_0^x = x e^{-x^2}}$$

$$\int_0^5 x e^{-x^2}\,dx =$$

Substitution

$u = x^2$

$du = 2x\,dx$

$du = 2x\,dx$

$$= \int e^{-u}\,\frac{du}{2}$$

$$= -\frac{e^{-u}}{2} = -\frac{e^{-x^2}}{2}$$

$$\int_0^5 -\frac{e^{-x^2}}{2}\,dx = -\frac{e^{-25}}{2} + \frac{1}{2}$$

$$= \boxed{\frac{1}{2} - \frac{e^{-25}}{2}}$$