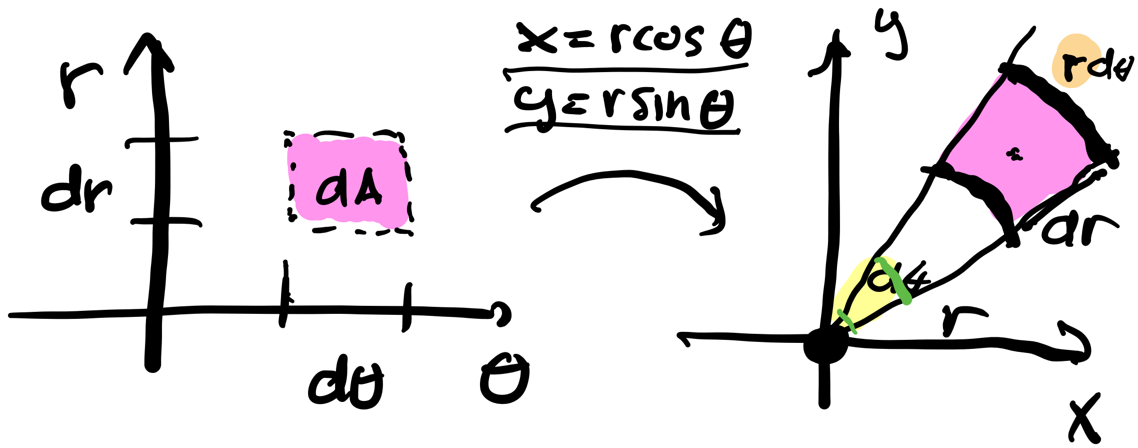
Unit 16

Surface integrals

1 Polar integrals

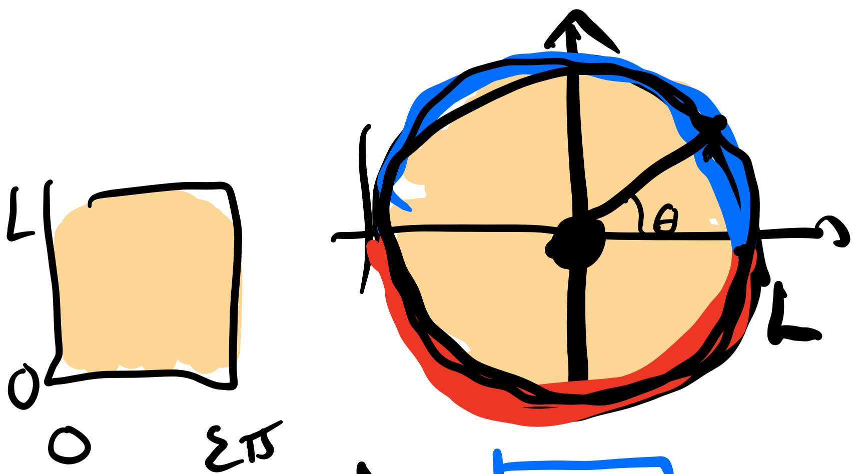


The area $dr\,d\theta$ gets deformed to $\boxed{r}\,dr\,d\theta$

If we switch to Polar coordinates, introduce a factor $\boxed{r}$.

$$dx\,dy \rightsquigarrow r\,dr\,d\theta$$

a) What is the area of a circle of radius L?



$$\int_{-L}^{L}\int_{\sqrt{L^2-x^2}}^{\sqrt{L^2-x^2}} 1 \; dy\, dx$$

Area

$$\int_{-L}^{L} 2\sqrt{L^2-x^2}\; dx$$

How would you do that?

Trig substitution?

Better: use Polar coordinates!

$$\int_0^{2\pi}\int_0^{L} 1\cdot \boxed{r}\, dr\, d\theta$$

Integration factor

$\iint_R 1\, dxy = \underline{\text{Area}}$

$\int_I 1\, dx = \underline{\text{length}}$

$$\int_0^{2\pi} \frac{L^2}{2}\, d\theta = \frac{2\pi L^2}{2} = \boxed{\pi L^2}$$

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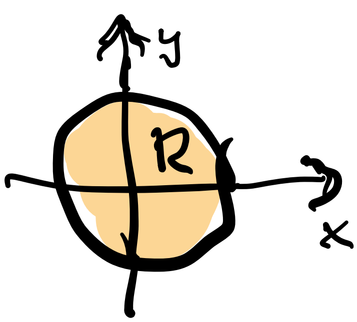
b) $I = \iint_R (x^2+y^2)\, \underbrace{dx\, dy}_{dA}$



Moment of Inertia.

$\frac{\omega^2}{2}\cdot I$ energy

$$\int_0^{2\pi} \left[\int_0^L r^2 \boxed{r}\, dr \right] d\theta$$

$$\int_0^{2\pi} 1\, d\theta = 2\pi$$

$$\int_0^{2\pi} \left(\frac{r^4}{4} \Big|_0^L \right) d\theta = \frac{L^4}{4} \cdot 2\pi$$

you read off the bound from the picture

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② Gifted

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

single variable problem

$$f(x, y)$$

$$\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy$$

turns out to be I^2

<u>Use Polar coordinates</u>:

$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,\boxed{r}\,dr\,d\theta$$

$$2\pi\left(\frac{-e^{-r^2}}{2}\Big|_0^{\infty}\right)$$

$$= 2\pi\left(\frac{-e^{-\infty^2}+e^{-0}}{2}\right)$$

$$= \boxed{\pi}$$

Now split into (1D) integrals
$e^{-x^2-y^2} = e^{-x^2}e^{-y^2}$

$$\pi = \int_{-\infty}^{\infty} \boxed{\int_{-\infty}^{\infty} e^{-x^2}\, dx}\; e^{-y^2}\, dy$$

$$I$$

$$I \cdot \boxed{\int_{-\infty}^{\infty} e^{-y^2}\, dy} = I^2$$

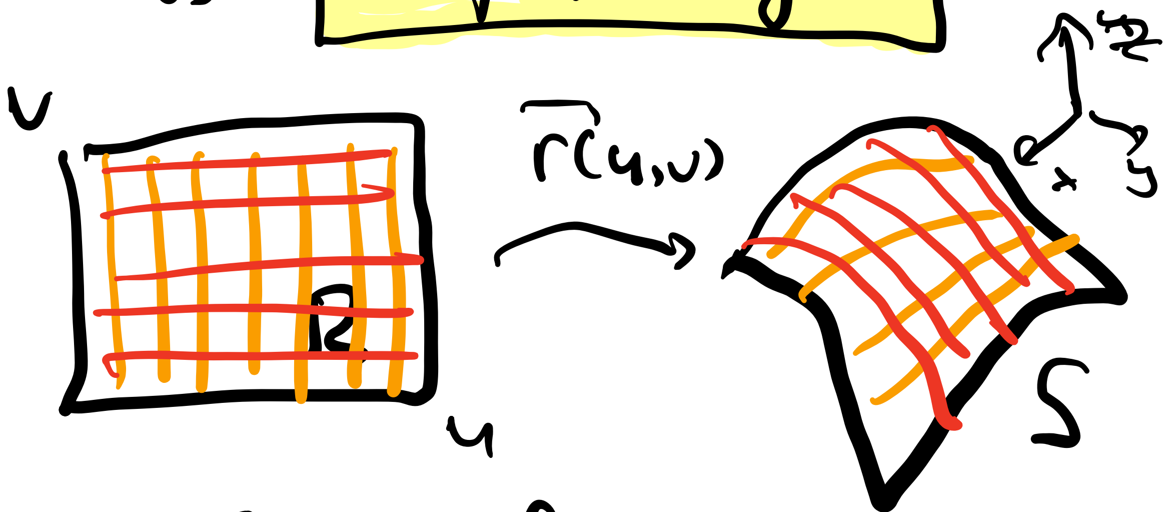
$$I \longrightarrow \boxed{I = \sqrt{\pi}}$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2}\, dx = 1$$

Important in stats!

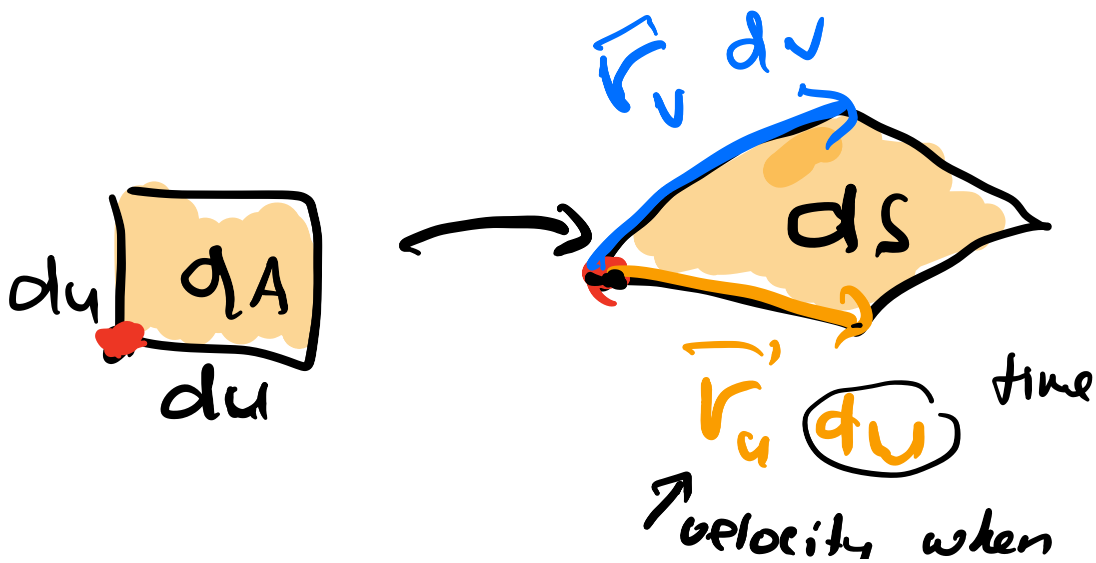
→ <u>Normal distribution</u>.

3) Surface integrals



The surface area of can be written as a double integral over R.

We have to understand the distortion factor



changing u

$$dS = \left| \vec{r}_u\, du \times \vec{r}_v\, dv \right|$$
$$= \left| \vec{r}_u \times \vec{r}_v \right| du\, dv$$

$$\boxed{\text{Surface area} = \iint_R \left| \vec{r}_u \times \vec{r}_v \right| du\, dv}$$

vector

magnitude

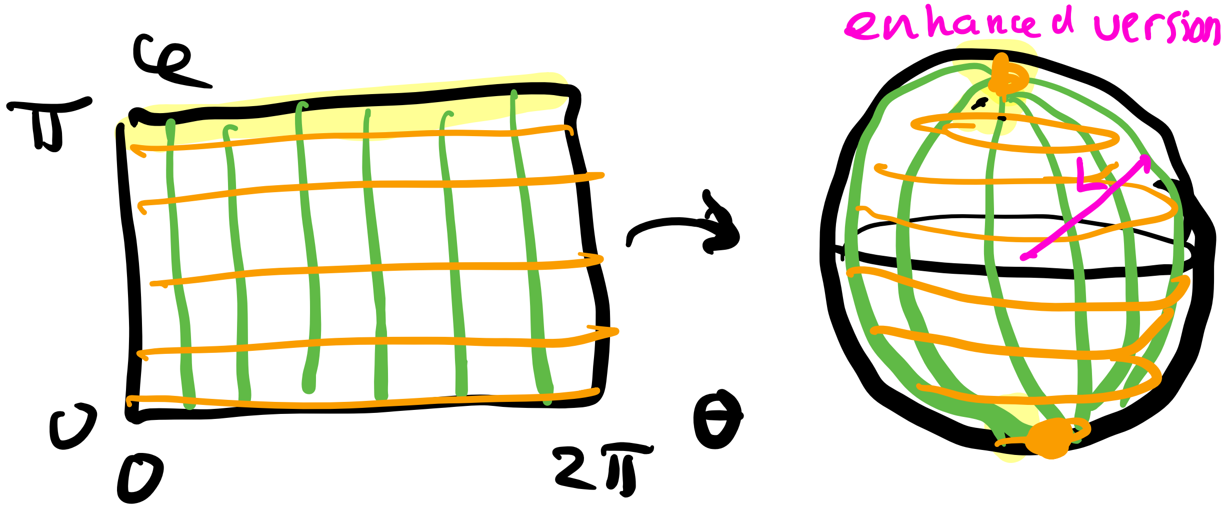
This reduces things to a double integral.

④ Sphere

radius 1

radius L

enhanced version



$$\vec{r}(\varphi,\theta) = \begin{bmatrix} L\sin\varphi\cos\theta \\ L\sin\varphi\sin\theta \\ L\cos\varphi \end{bmatrix}$$

✓ brute force

$$\vec{r}_\varphi = \begin{bmatrix} L\cos\varphi\cos\theta \\ L\cos\varphi\sin\theta \\ L-\sin\varphi \end{bmatrix}$$

do this on your own again

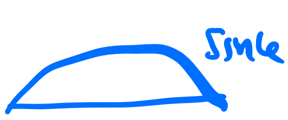
$$\vec{r}_\theta = \begin{bmatrix} L-\sin\varphi\sin\theta \\ L\sin\varphi\cos\theta \\ L \quad 0 \end{bmatrix}$$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{bmatrix} L^2\sin\varphi\sin\varphi\cos\theta \\ L^2\sin\varphi\sin\varphi\sin\theta \\ L^2\sin\varphi\cos\varphi \end{bmatrix}$$

$$\boxed{\begin{matrix} |a\cdot\vec{v}| \\ |a|\cdot|\vec{v}| \end{matrix}}$$

$$= L^2\sin\varphi \begin{bmatrix} \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \\ \cos\varphi \end{bmatrix}$$

$$|\vec{r}_\varphi \times \vec{r}_\theta| = L^2\sin\varphi \left\| \begin{bmatrix} \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \\ \cos\varphi \end{bmatrix} \right\|$$



$$= L^2\sin\varphi$$

$\uparrow$ 1

0 π

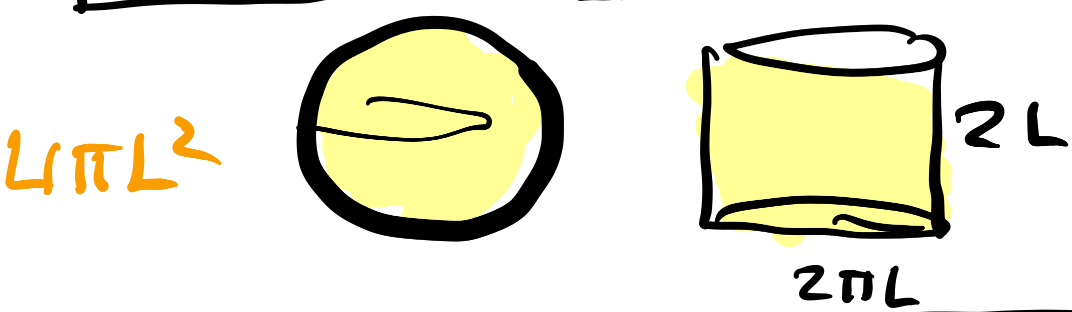
The distortion factor for the sphere is $L^2 \sin\varphi$

The surface area of the sphere of radius L is

$$\int_0^{2\pi} \boxed{\int_0^{\pi} L^2 \sin\varphi \, d\varphi} \, d\theta$$

$$2L^2 = \boxed{4\pi L^2}$$

Archimedes



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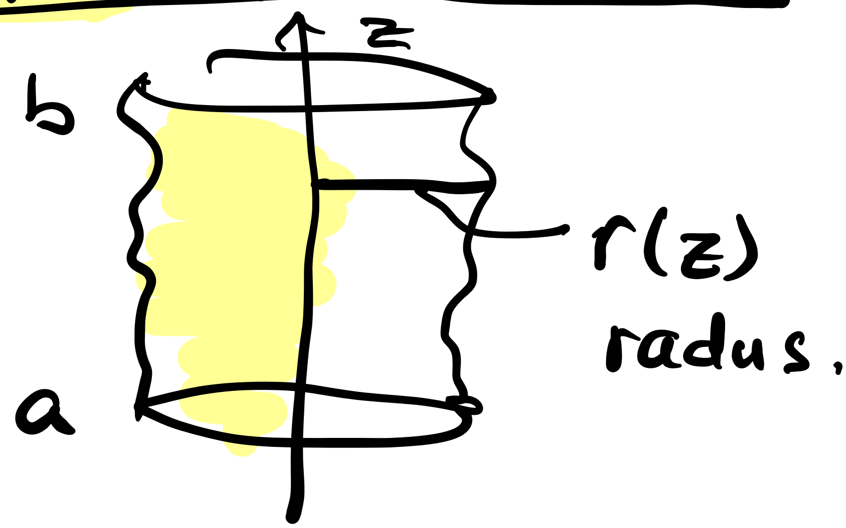
(5) Surface of revolution



$$\vec{r}(\theta, z) = \begin{bmatrix} r(z)\cos\theta \\ r(z)\sin\theta \\ z \end{bmatrix}$$

$$|\vec{r}_\theta \times \vec{r}_z| = r(z)\sqrt{1 + r'(z)^2}$$

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distortion factor of surface of revolution

Surface area: $\boxed{2\pi \int_a^b r\sqrt{1+r'(z)^2}\, dz}$

Example:

Gabriel's trumpet

$r(z) = \frac{1}{z}$

Volume of the trumpet is finite and given by $\boxed{\pi}$

$$2\pi \int_1^\infty \frac{1}{z} \overbrace{\sqrt{1+\left(\frac{1}{z^2}\right)^2}}^{\geq 1}\, dz$$

integrate over θ

$$\geq \int_1^\infty \frac{1}{z}\, dz$$

$$\int_1^b \frac{1}{z}\,dz = \log(z)\Big|_1^b$$

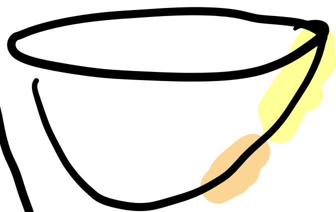
$$= \log b$$

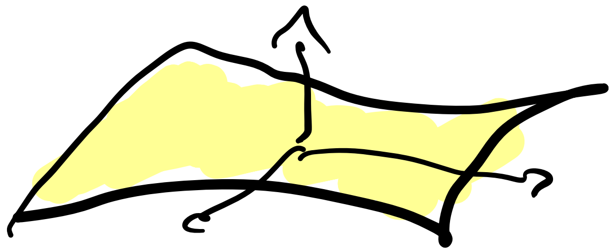
$$\downarrow$$

$$\infty$$

The trumpet has ∞ area but finite volume. Fill it with paint. But you can not paint it.

graph : $z = f(x,y)$





$\sqrt{1+f_x^2+f_y^2}$ distortion factor