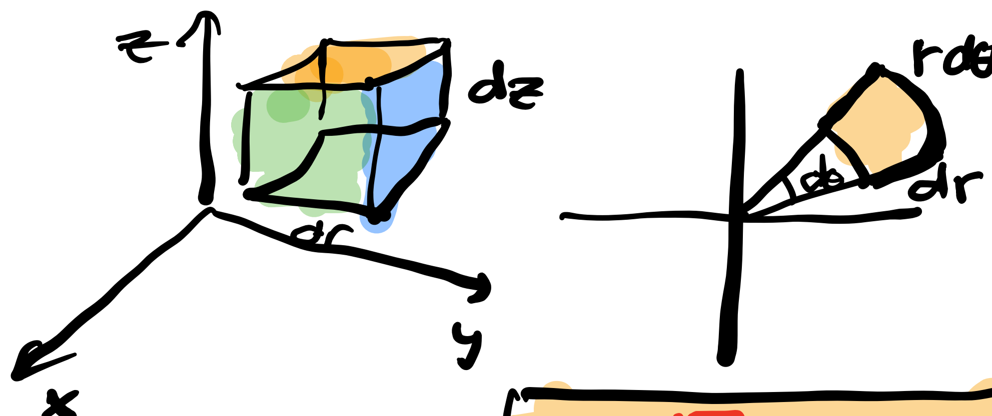
Unit 18

Spherical integration

1) Cylindrical coordination

Polar coordinates in x, y
z coordinate is not changed.

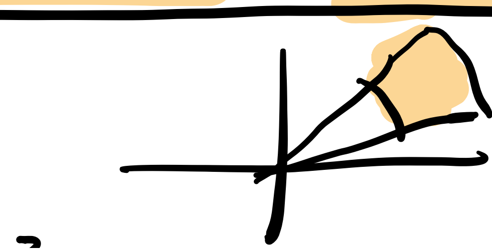


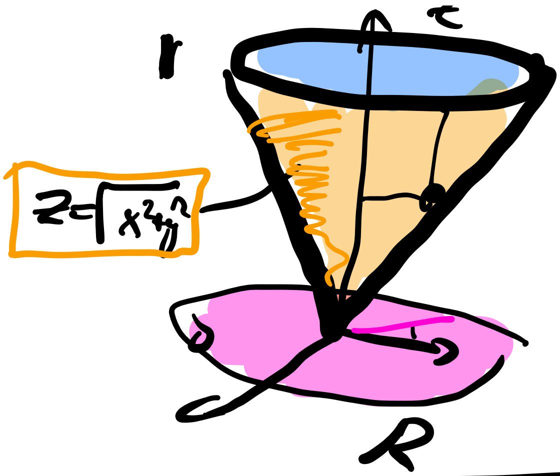
Volume element

$$dV = \boxed{r}\, dr\, d\theta\, dz$$

This is analog to

$$dA = \boxed{r}\, dr\, d\theta$$

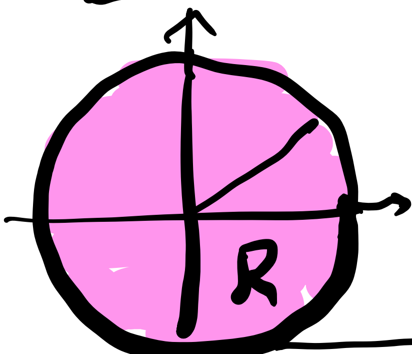




Volume of a cone

$$\iint_R \int_{\sqrt{x^2+y^2}}^{1} 1 \, dz\, dx\, dy$$

$$\iint_R \left(1-\sqrt{x^2+y^2}\right) dx\, dy$$



Switch to polar coordinates!

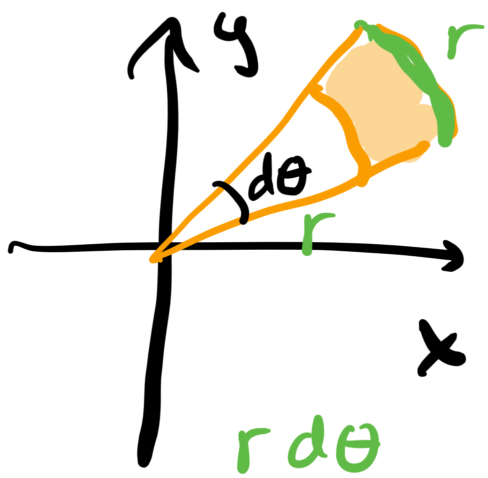
$$\int_0^{2\pi}\int_0^{1} (1-r)\, r \, dr\, d\theta$$

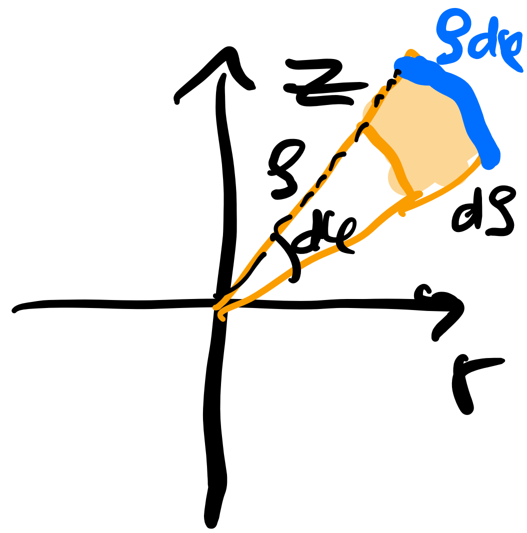
$$\int_0^{2\pi} \frac{r^2}{2}-\frac{r^3}{3}\Big|_0^1 \, d\theta$$

$$2\pi \cdot \frac{1}{6} = \boxed{\frac{\pi}{3}}$$

(Area of R · height / 3 .)

2) Spherical coordinates



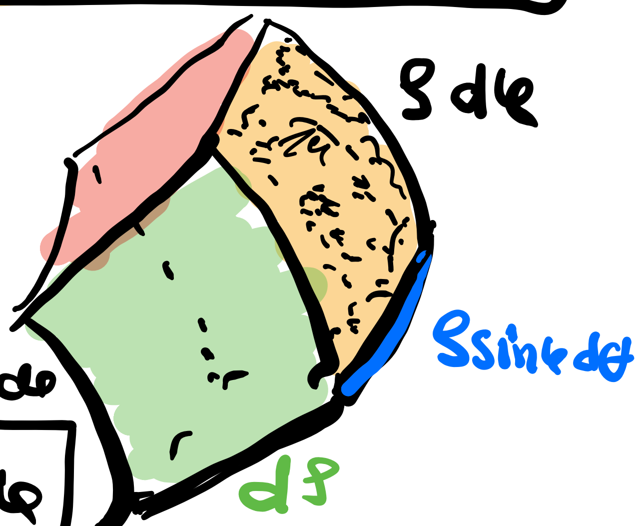


$r\,d\theta$

$\rho \sin\varphi \, d\theta$

$$\boxed{\begin{aligned} dV &= \rho\sin\varphi\, d\theta \; \rho\, d\varphi \;\; d\rho \\ &= \boxed{\rho^2 \sin\varphi} \; d\varphi\, d\theta\, d\rho \end{aligned}}$$

When going to spherical coordinates, include the factor $\boxed{\rho^2 \sin\varphi}$



3) **Sphere computation**

Sphere of radius L

$$x^2+y^2+z^2 \leq L^2$$

Do this in spherical coordinates!

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{L} 1 \; \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

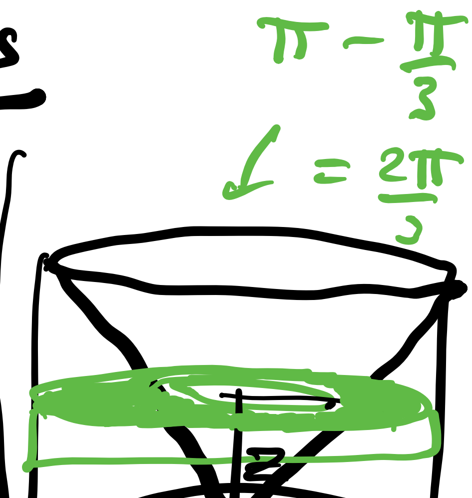
$$\rho^2 \sin\varphi = \left|\vec{r}_\theta \times \vec{r}_\varphi\right|$$

when parametrising the sphere

$$\int_0^{2\pi}\underbrace{\left[\int_0^{\pi}\frac{L^3}{3}\sin\varphi\,d\varphi\right]}_{\frac{2L^3}{3}}d\theta$$

$$= 2\pi\cdot\frac{2L^3}{3} = \boxed{\frac{4\pi L^3}{3}}$$

Archimedes

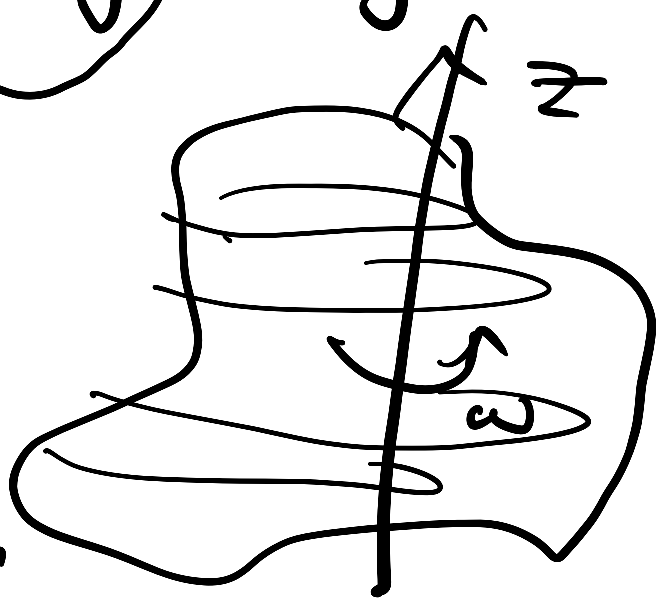


$$\int_0^1 r^2\pi\,dz = \int_0^1 (1-z^2)\,dz \qquad \Big| \qquad \int_0^1 1-z^2\,dz$$

Moment of inertia.

$$r^2 = \rho^2 \sin^2\phi$$

$$I = \iiint_E (x^2+y^2)\, dx\,dy\,dz$$



$$\frac{\omega^2}{2} I$$

$=$

Energy of the rotating solid.

For the sphere of radius L, what is the moment of inertia.

$I =$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{L} \rho^2 \sin^2\varphi \; \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$\boxed{\int_0^{2\pi}\int_0^{\pi}\int_0^{L} \rho^4 \sin^3\varphi \, d\rho \, d\varphi \, d\theta}$$

$r = \rho \sin\varphi$

$r^2 = \rho^2 \sin^2\varphi$

$\parallel$

$x^2 + y^2$

$$\boxed{\frac{L^5}{5}\int_0^{2\pi}\int_0^{\pi} \sin^3\varphi \, d\varphi \, d\theta}$$

$$\frac{L^5}{5}\int_0^{2\pi}\int_0^{\pi} \sin\varphi\,(1-\cos^2\varphi)\, d\varphi\, d\theta$$

$$\frac{L^5}{5}\int_0^{2\pi}\int_0^{\pi} \sin\varphi \, d\varphi\, d\theta = 2$$

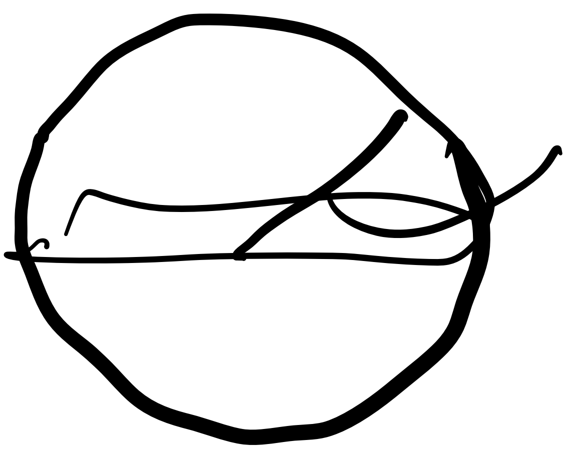
$$u = \cos\varphi$$
$$du = -\sin\varphi\, d\varphi$$

$$\frac{L^5}{5}\int_0^{2\pi}\int_0^{\pi} \sin\varphi \cos^2\varphi \, d\varphi\, d\theta$$

$$\frac{L^5}{5}\cdot 4\pi + \frac{L^5}{5}\,\frac{\cos^3\varphi}{3}\Big|_0^{\pi}\cdot 2\pi$$

$$\frac{L^5}{5}\left(4\pi - \frac{4\pi}{3}\right) = \boxed{\frac{8\pi L^5}{5}}$$

$$\boxed{\frac{8\pi L^5}{15}} = I$$



$L = 6000\text{km}$

$\circlearrowleft\ \frac{1}{1\,\text{day}} = \omega$

$$= \frac{\quad}{24 \cdot 3600}$$

$$\text{Energy} = \frac{\omega^2}{2} \cdot I$$

$$\frac{d}{d\rho}\boxed{\frac{4\pi\rho^3}{3}}$$

$$= \boxed{4\pi\rho^2}$$