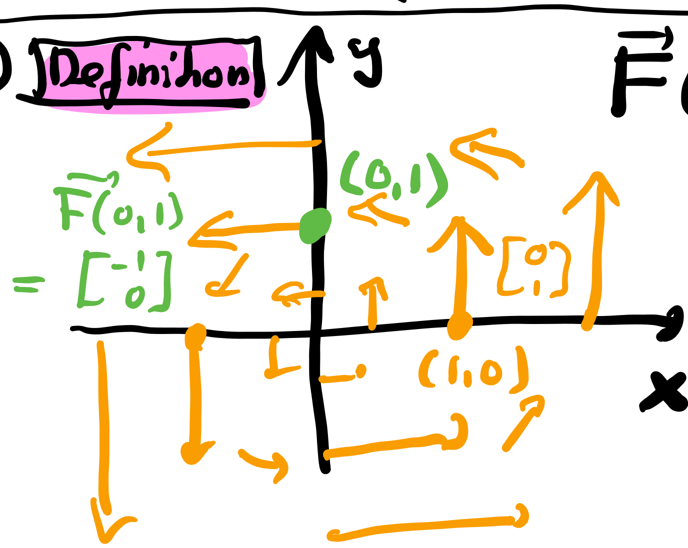
Unit 19 Vector fields

1) Definition

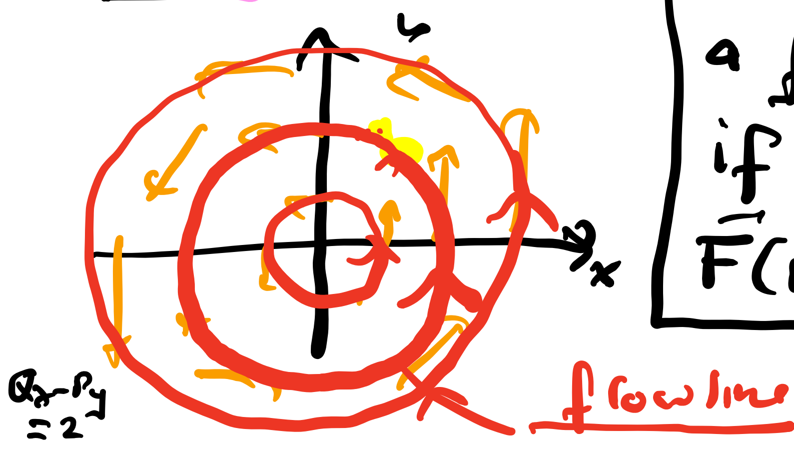


$$\vec{F}(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

$x=1$, $y=0$ → $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

vector valued function of 2 variables.

2) Flow Lines



$\vec{r}(t)$ is called a flow line if

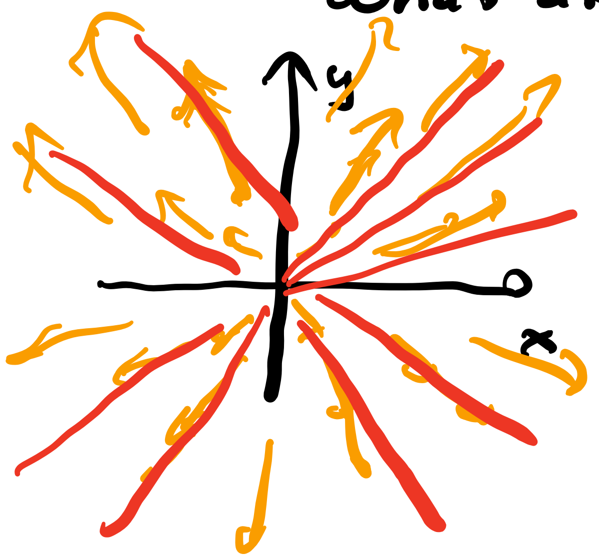
$$\vec{F}(\vec{r}(t)) = \vec{r}\,'(t)$$

$$\vec{r}(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\vec{r}\,'(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} = \vec{F}(x,y)$$

E: $\vec{F}(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

what are the flow lines?



$Q_x - P_y = 0$

Mathematica:

VectorPlot

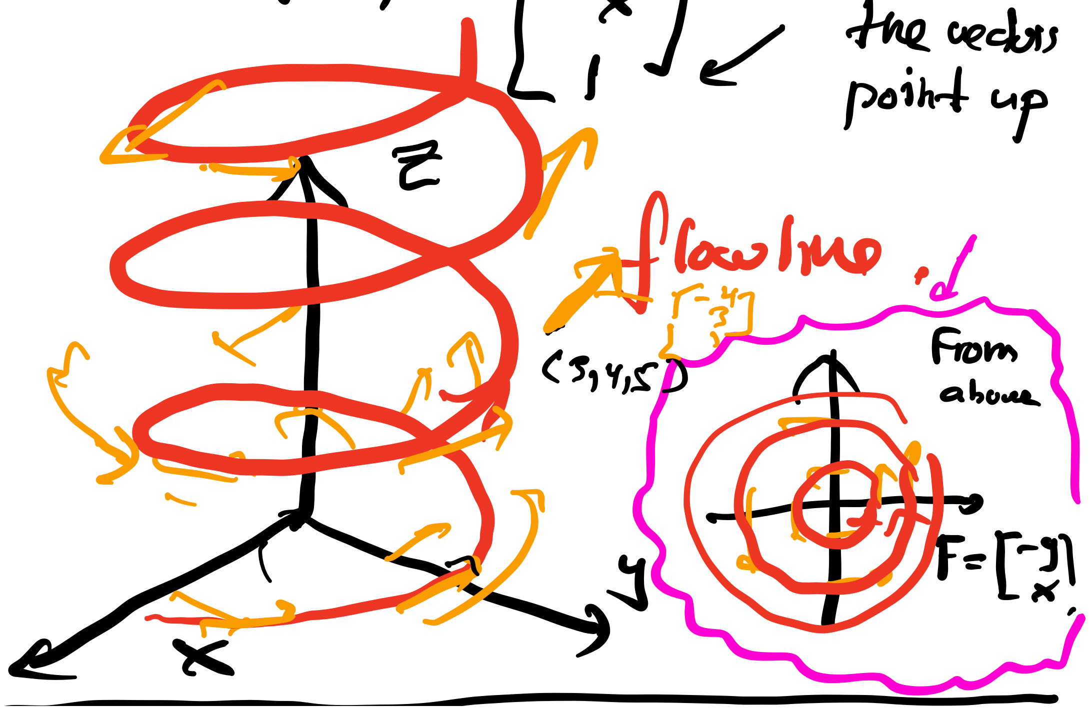
Better:

StreamPlot

also draws the flow lines.

F: $\vec{F}(x,y,z) = \begin{bmatrix} -y \\ x \\ 1 \end{bmatrix}$

the vectors point up



3. Gradient fields

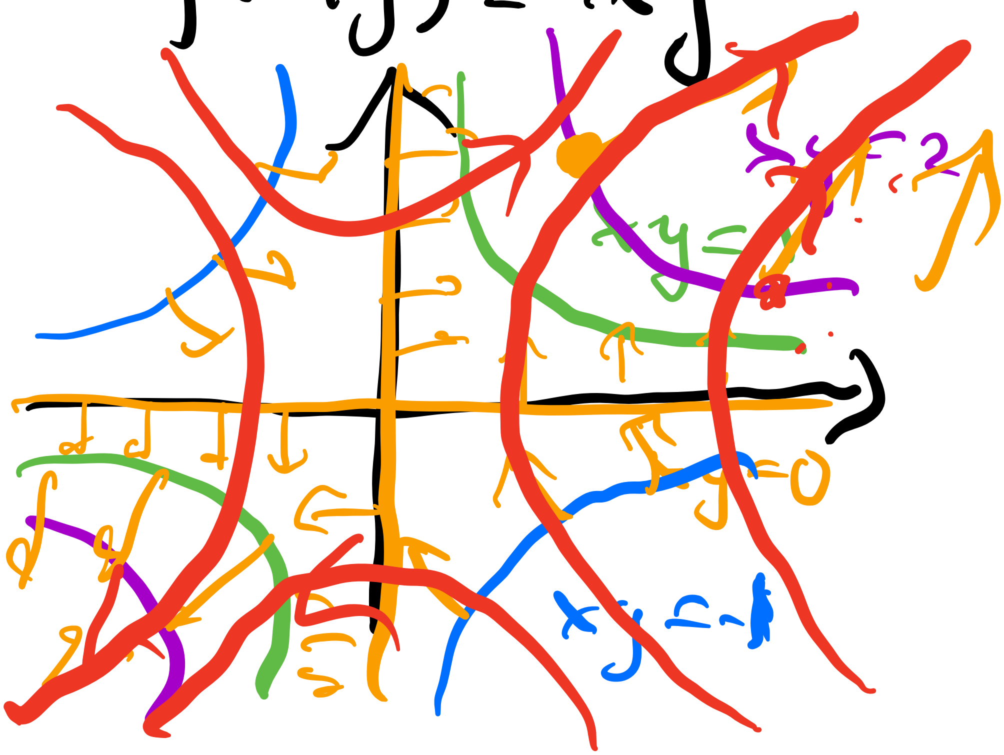
If $\vec{F} = \nabla f$ for a function f, then $\vec{F} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$ is called a <u>gradient field</u>. f is called <u>potential</u>

<u>Example:</u>

$$\vec{F}(x,y) = \begin{bmatrix} y \\ x \end{bmatrix}$$

what function f does this? $= \begin{bmatrix} f_x \\ f_y \end{bmatrix}$

$f(x,y) = xy$ works,



"Angels dance upwards"

∇f points upward.

Problem

$$\boxed{\begin{array}{l}\text{Find } f(x,y) \\ \text{such that} \\ \vec{F}(x,y) = \begin{bmatrix} -y \\ x \end{bmatrix} = \nabla f\end{array}}$$

There is none!

$$\begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} P \\ Q \end{bmatrix}$$

$$P_y = f_{xy}$$
$$Q_x = f_{yx}$$

equal

This means

$$\boxed{Q_x - P_y = 0}$$

Clairaut test

(Next week, we call

$\text{curl}(\vec{F}) = Q_x - P_y$)

If $Q_x - P_y \neq 0$ somewhere, then $\vec{F}$ can not be a gradient field.

If $Q_x - P_y = 0$ then there is a $f(x,y)$

How do we find f

Example $= \begin{bmatrix} P \\ Q \end{bmatrix} =$

$$\vec{F}(x,y) = \begin{bmatrix} 3y + x^2 + \sin x \\ y^3 + 3x + e^y \end{bmatrix}$$

Find f !

Clairaud : $Q_x = 3$

$P_y = 3$ ✓

$$\begin{bmatrix} 3y + x^2 + \sin x \\ y^3 + 3x + e^y \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

A Integrate the first equation

$$3xy + \frac{x^3}{3} - \cos x + \boxed{C(y)} = f$$

B Now differentiate w.r.t. y

$$\cancel{3x} + C'(y) = y^3 + \cancel{3x} + e^y$$

C Find $C(y)$ ...

So, $C(y) = \frac{y^4}{4} + e^y$

$$\boxed{f = 3xy + \frac{x^3}{3} - \cos x + \frac{y^4}{4} + e^y}$$

$+ C$

possible, but we need one function so we choose $C = 0$

Skills: . match vector fields
. Find Potentials or see that there is none.