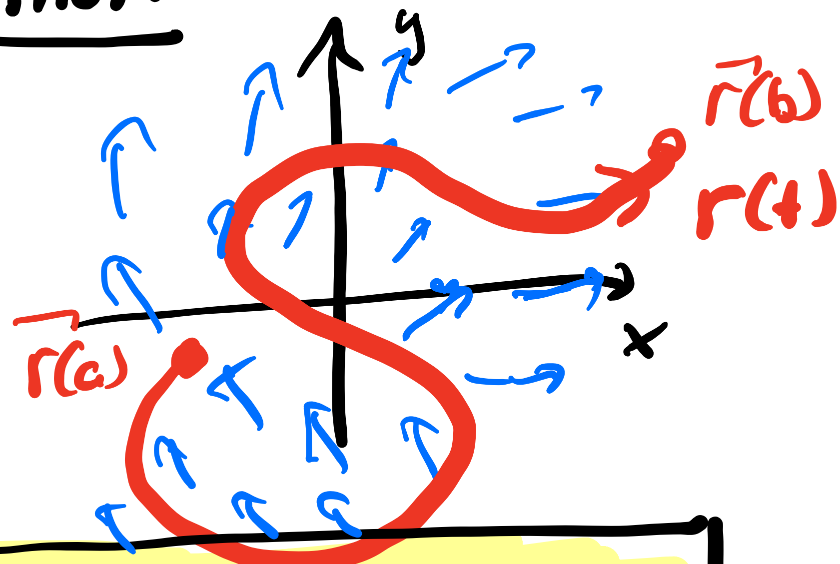
# Unit 20

# Line integrals

Force

$\vec{F}(x,y)$

## ① Definition




$$\int_C \vec{F}\cdot d\vec{r}$$

$$\int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

is called the line integral of $\vec{F}$ along the curve $C : \vec{r}(t)$

Force · Velocity = <u>Power</u>

$$\int_a^b \text{Power}(t)\,dt = \underline{\text{Work}}$$

The interpretation is that

$\int_c \vec{F}\cdot dr$ is the work of $\vec{F}$

done on the body moving along the path

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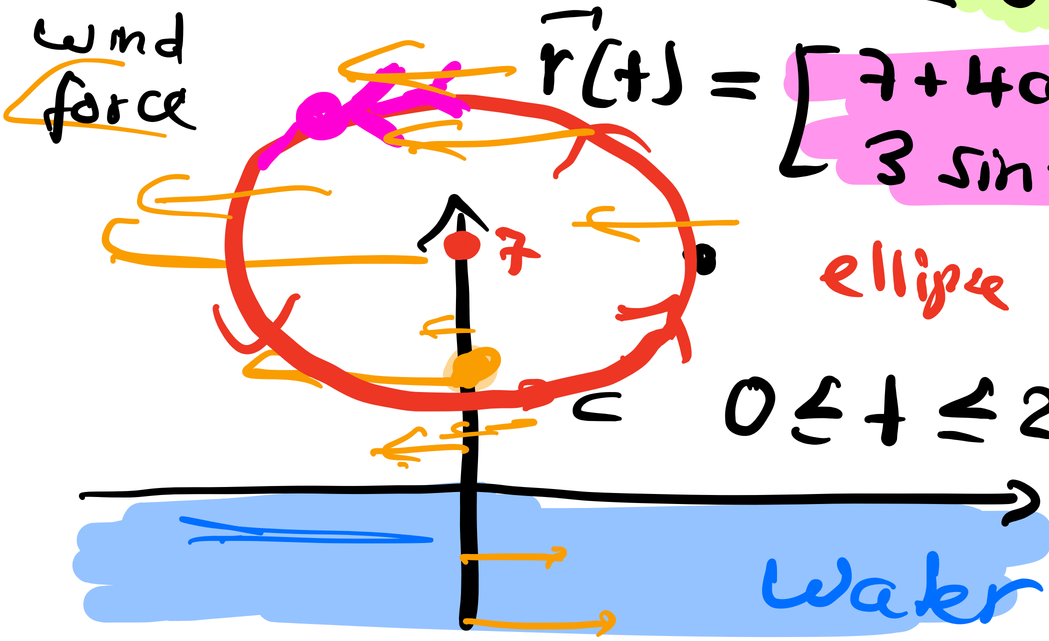
② <u>Example</u>: $\vec{F}(x,y) = \begin{bmatrix} -y \\ 0 \end{bmatrix}$

$\vec{r}(t) = \begin{bmatrix} 7 + 4\cos t \\ 3\sin t \end{bmatrix}$




ellipse $= \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

$0 \le t \le 2\pi$

Intuition: is $\int_C \vec{F} \cdot d\vec{r}$ positive or negative

It should be positive : gain more up then lose near water

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$$\int_0^{2\pi} \begin{bmatrix} -3\sin t \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -4\sin t \\ 3\cos t \end{bmatrix} dt$$

$$\int_0^{2\pi} 12 \sin^2 t \, dt$$

$$= \int_0^{2\pi} 12 \left( \frac{1-\cos 2t}{2} \right) dt$$

$$= 6 \int_0^{2\pi} (1 - \cos 2t) dt$$

$$= \boxed{12\pi}$$

Indeed the gained energy is positive.

~~Don't use~~

$$\cancel{\int \vec F \cdot d\vec r = \int P\,dx + Q\,dy}$$

$$\cancel{P \cdot \frac{dx}{dt}\,dt + Q\,\frac{dy}{dt}\,dt}$$

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## (3) Fundamental theorem of Line integrals

If $\vec F = \nabla f$, then the line integral can be computed quickly

**Theorem** (FTLI)

$$\int_a^b \nabla f(\vec r(t)) \cdot \vec r\,'(t)\,dt = f(\vec r(b)) - f(\vec r(a))$$

$$\frac{d}{dt} f(\vec{r}(t))$$ chain rule

Proof of the theorem follows from the FTC

Fundam. thm of calc.

$$\boxed{\text{First generalization of the FTC to higher dimension}}$$

Example:

$$\vec{F} = \begin{bmatrix} e^{e^{e^{x}}} \\ \sin \log y \end{bmatrix} = \begin{bmatrix} P \\ Q \end{bmatrix}$$

$$\vec{r}(t) = \begin{bmatrix} \cos t + \sin(100t)/1000 \\ \sin(200t) \end{bmatrix}$$

$0 \le t \le 2\pi$

What is $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ ?

$$\int_0^{2\pi} \begin{bmatrix} e^{e^{e^{\cos t + \sin 100t/100t}}} \\ \sin \log(\sin 200t) \end{bmatrix} \cdot \begin{bmatrix} \cdots \\ \cdots \end{bmatrix} dt$$

This is zero because

$\vec{F}$ is a gradient field $\vec{F} = \nabla f$

$\vec{r}(0) = \vec{r}(2\pi)$ closed curve

By the theorem

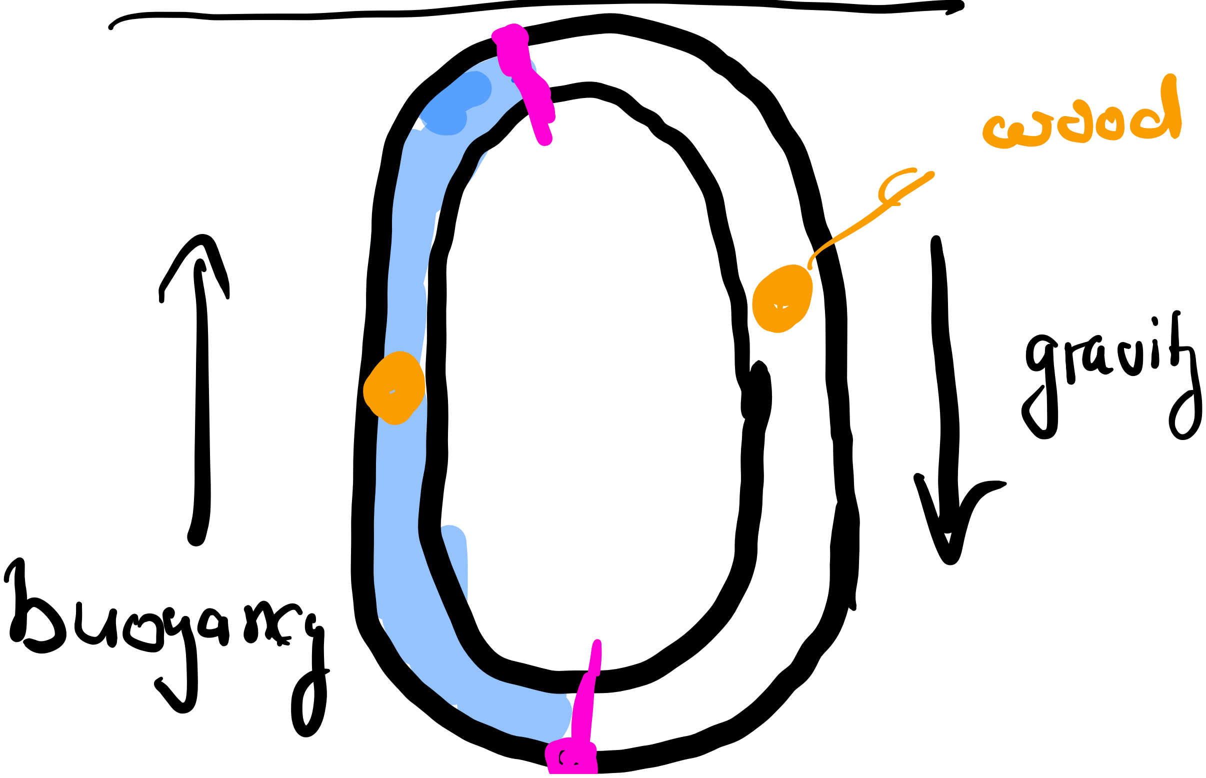
$$f(\vec{r}(2\pi)) - f(\vec{r}(0)) = 0$$

Closed loop property

= Conservation of energy.

All fundamental fields in nature are gradient field.

Here is an idea :



# Perpetual motion machine

Energy out of nothing