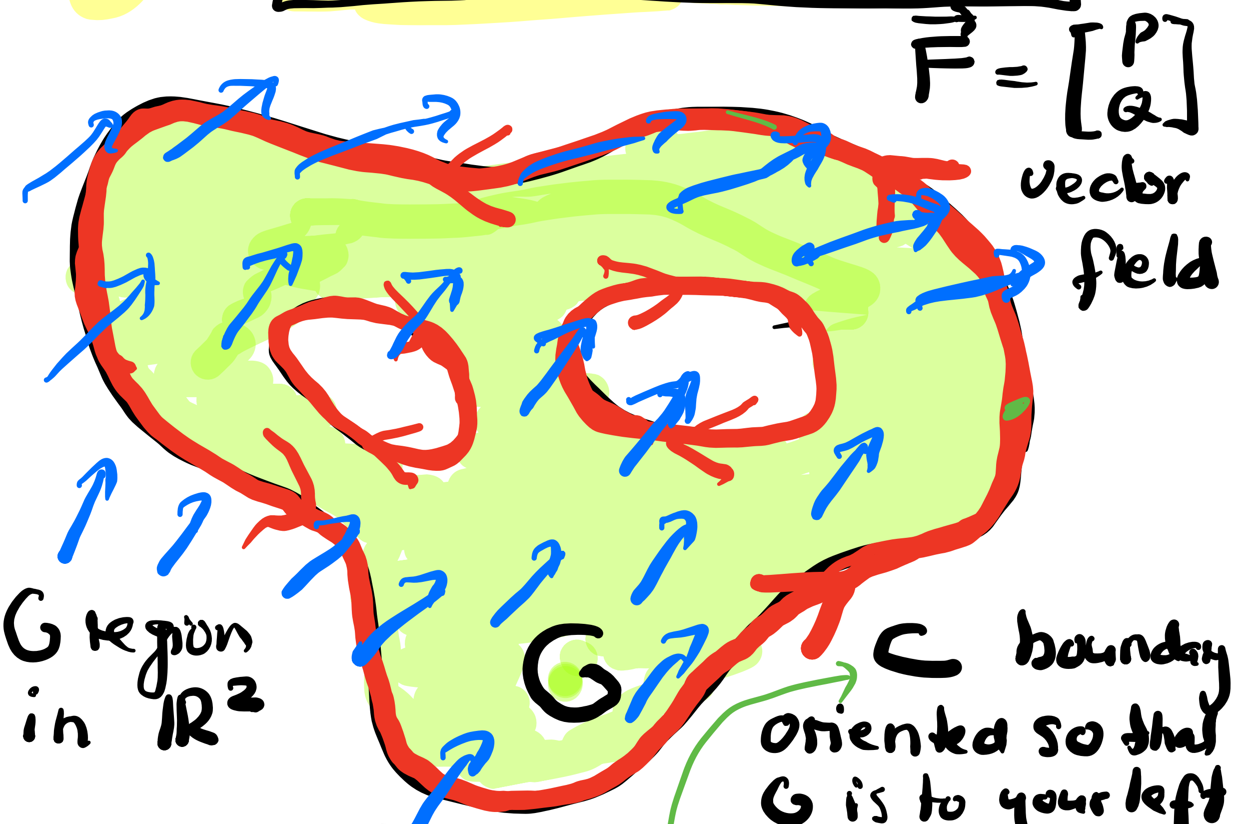
Unit 21

we are in Flatland

① Green's Theorem



Theorem:

$$\iint_G \text{curl}(\vec{F})\,dA = \int_C \vec{F} \cdot d\vec{r}$$

Reminders : $\text{curl}\left(\begin{bmatrix} P \\ Q \end{bmatrix}\right) = Q_x - P_y$

$\longrightarrow$ unit 22 curl of $\vec{F}$

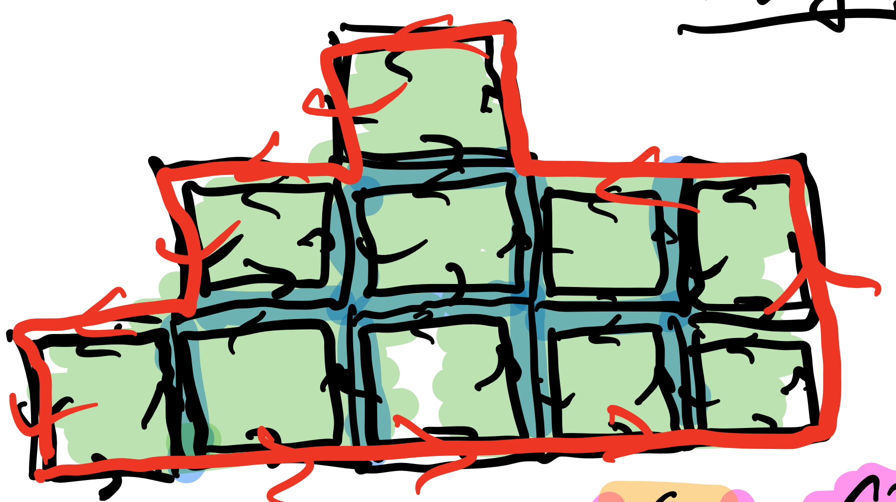
$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

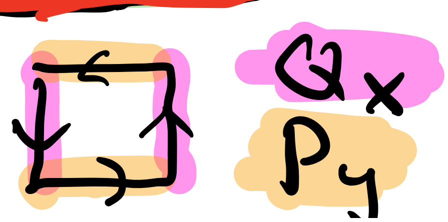
If there are several components of C , we add all these line integrals.

Why is it true!

Magic

intuition





$Q_x - P_y$

If we add up all these little line integrals then is a cancellation.
The interior parts cancel away! Only the boundary part survives

2 2 important cases and an example

A $\boxed{\text{Curl}(\vec{F}) = 0 \text{ everywhere}}$

"irrotational"
This has appeared as a Clairaut test for
$$\vec{F} = \nabla f.$$

If $\text{curl}(\vec{F}) = 0$ everywhere, then $\vec{F} = \nabla f$!

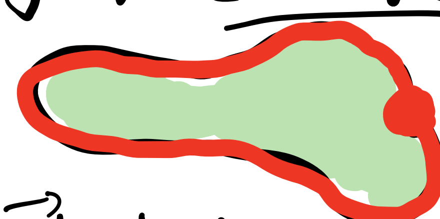
side remark: This assumes $\vec{F}$ and $\text{curl}(\vec{F})$ to be continuous everywhere (*)

$$\iint_G \text{curl}(\vec{F})\, dA = 0$$

We know already

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

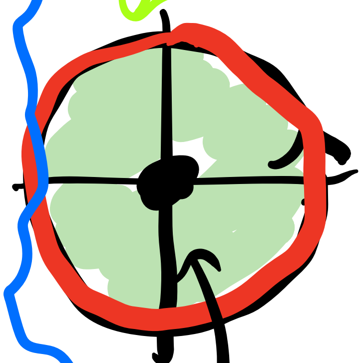
because of the FTLI



$$\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = f(\vec{r}(b)) - f(\vec{r}(a)) = 0$$

(*) $\vec{F} = \begin{bmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{bmatrix}$ is not defined everywhere but still $\text{curl}(\vec{F}) = 0$ everywhere except at $(0,0)$

not simply connected



$\text{curl}(F) = 0$ $\quad \int_C \vec{F} \cdot dr = 2\pi$

(B) $\boxed{\text{curl}(\vec{F}) \equiv 1}$ everywhere

This is important because

$$\iint_G \text{curl}(F)\, dA$$

is the <u>area</u> of G.

→ Planimeter

analog computer is based on that?

We use a simple vector field which work:

$$\vec{F} = \begin{bmatrix} 0 \\ x \end{bmatrix} = \begin{bmatrix} P \\ Q \end{bmatrix}$$

$$\vec{F} = \begin{bmatrix} -y \\ 0 \end{bmatrix}$$

C) An example

$$\vec{F} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

flow lines

$\text{curl}(\vec{F}) = 2$

Disk of radius 2

G region bound by $\begin{bmatrix} x \\ y \end{bmatrix} = \vec{r}(t) = \begin{bmatrix} 2+2\cos t \\ 2\sin t \end{bmatrix}, \quad 0 \le t < 2\pi$

LHS in Green:

left hand side LHS

$$\iint_G \underbrace{\mathrm{curl}(F)}_{2}\, dA = 2 \cdot \mathrm{Area}(G)$$
$$= 2 \cdot 4\pi$$
$$= \boxed{8\pi}$$

RHS in Green:

$\vec{r}\,'(t)$

$$\int_0^{2\pi} \begin{bmatrix} -2\sin t \\ 2+2\cos t \end{bmatrix} \cdot \begin{bmatrix} -2\sin t \\ 2\cos t \end{bmatrix} dt$$

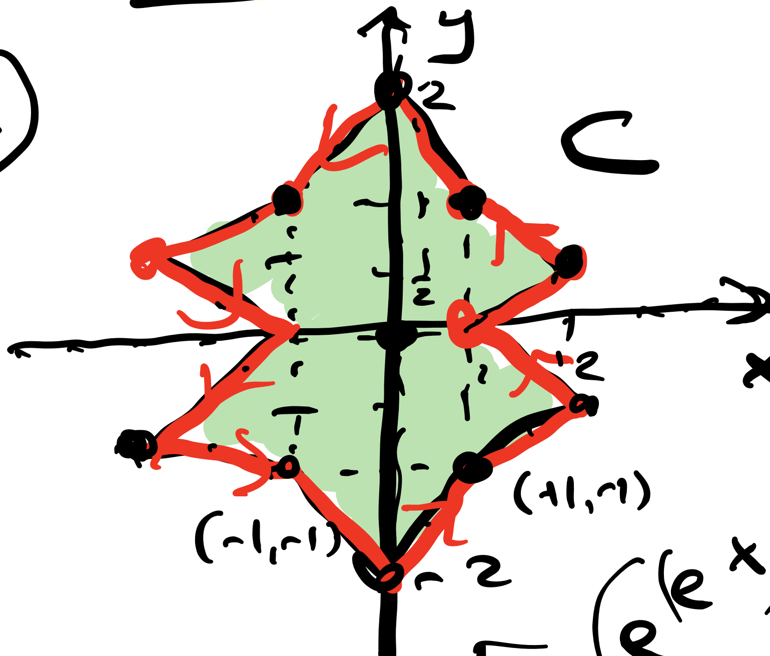
$$= \int_0^{2\pi} 4\sin^2 t + 4\cos t + 4\cos^2 t \, dt$$

$$= \int_0^{2\pi} 4\cos t + 4 \, dt$$

$$= 4\sin t \Big|_0^{2\pi} + 4 \cdot 2\pi = \boxed{8\pi}$$

③ 2 Examples

Ⓐ



Find

$$\int_C \vec{F} \cdot d\vec{r}$$

where

$$\vec{F} = \begin{bmatrix} e^{e^{(e^{x})}} + 5y \\ x - \sin\sin(y) \end{bmatrix}$$

You can not do this line integral directly.

Use the theorem:

-32

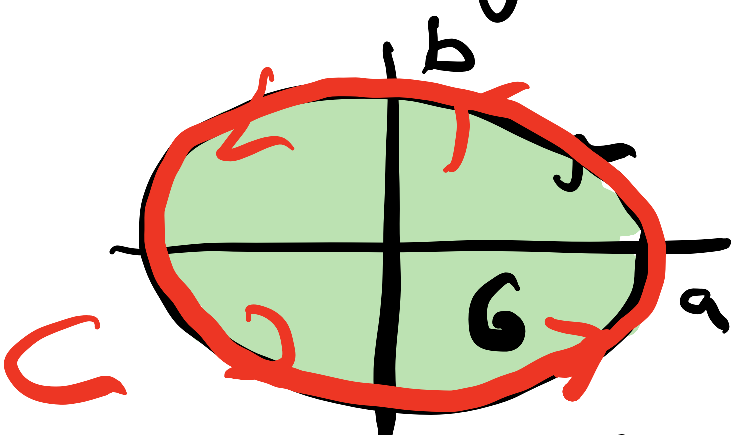
With the theorem, this is easy. Why can one see immediatly?

$$Q_x - P_y = -4$$

$$\iint_G \operatorname{curl}(\vec{F})\, dA = (-4)\ \text{Area of } (G)$$

$$= (-4)\cdot 8 = -32$$

B Hardest problem in geometry



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$$

$$\iint_G 1\, dA$$

we can use green's theorem

$\vec{F} = \begin{bmatrix} -y \\ x \end{bmatrix} / 2$ is possible

$\operatorname{curl}(\vec{F}) = 1$ usually take $\begin{bmatrix} 0 \\ x \end{bmatrix}$

we have to parametrize C

$$\vec{r}(t) = \begin{bmatrix} a\cos t \\ b\sin t \end{bmatrix}, \quad r'(t) = \begin{bmatrix} -a\sin t \\ b\cos t \end{bmatrix}$$

$$\int_0^{2\pi} \underline{\begin{bmatrix} -b\sin t \\ a\cos t \end{bmatrix}} \cdot \begin{bmatrix} -a\sin t \\ b\cos t \end{bmatrix} dt$$

$$= \int_0^{2\pi} \frac{ab}{2}\left(\sin^2 t + \cos^2 t\right) dt$$

$$= \frac{ab}{2} \cdot 2\pi$$

Direct: $$\int_{-a}^{a} \int_{-\sqrt{1-\frac{x^2}{a^2}}}^{\sqrt{1-\frac{x^2}{a^2}}} 1 \, dy \, dx$$

trig substitution:

$$= \int_{-a}^{a} 2\sqrt{1-\frac{x^2}{a^2}} \, dx$$

$\frac{x}{a} = \sin u$

$dx \frac{1}{a} = \cos u \, du$

7 minute break

10^{05}

restart w. unit 21