Unit 24

Divergence theorem

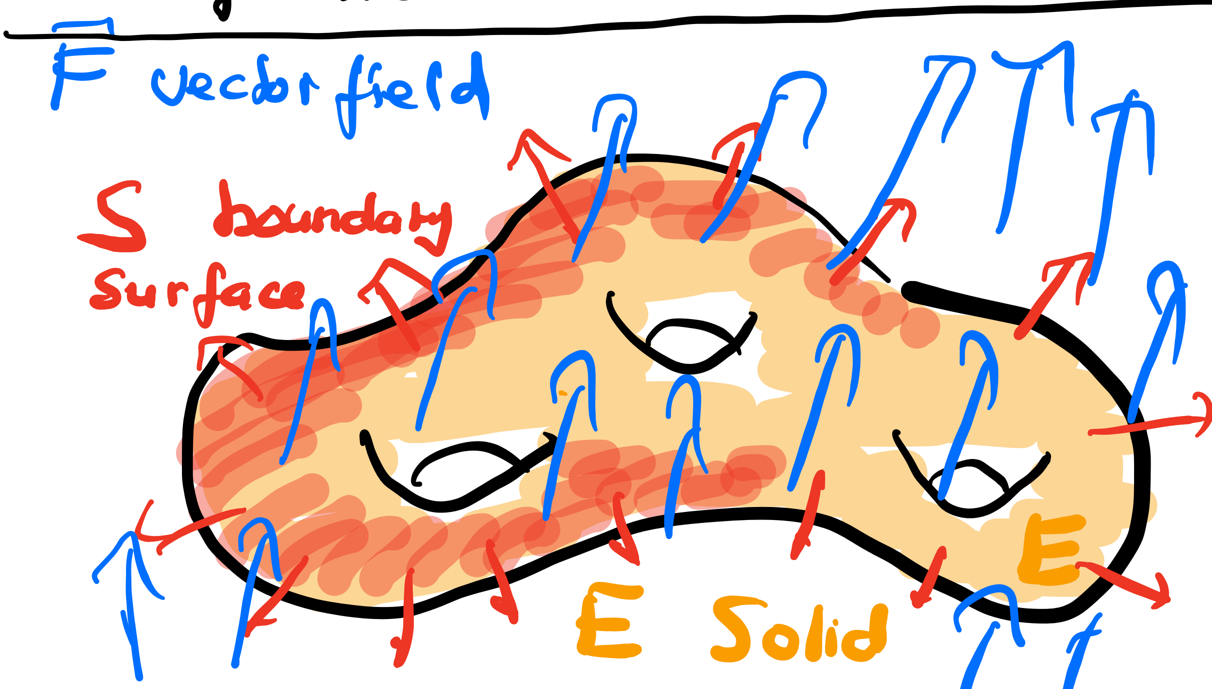
Theorem

$$\boxed{\iiint_E \operatorname{div}(\vec{F})\,dV = \iint_S \vec{F}\cdot d\vec{S}}$$

triple integral — flux integral

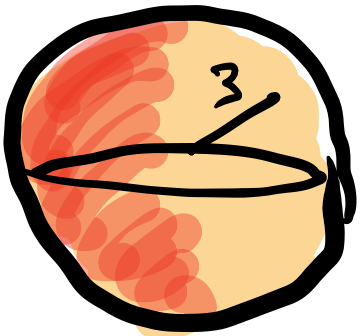
LHS: How much field is generated inside

RHS: How much field leaves the surface



S is oriented outwards

② **Example**



E: $x^2+y^2+z^2 \leq 9$
solid ball of radius 3

S: $x^2+y^2+z^2=9$
sphere of radius 3

$$\vec{F} = \begin{bmatrix} 0 \\ 1 \\ z \end{bmatrix} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

First: LHS of the theorem

$$\operatorname{div}(\vec{F}) = P_x + Q_y + R_z = 1$$

$$\iiint_E \operatorname{div}(\vec{F})\, dV = \iiint_E 1\, dV$$

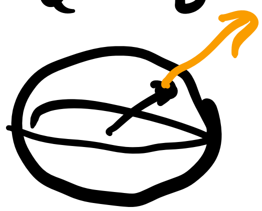
$$= \text{Volume of } (E) =$$

$$= \frac{4\pi}{3} 3^3 = \boxed{36\pi}$$

Second: RHS of the theorem

$$\vec{r}(\varphi,\theta) = 3\begin{bmatrix}\sin\varphi\cos\theta\\ \sin\varphi\sin\theta\\ \cos\varphi\end{bmatrix} = \begin{bmatrix}x\\ y\\ z\end{bmatrix} \qquad \begin{matrix}0\le\varphi\le\pi\\ 0\le\theta<2\pi\end{matrix}$$

$$\vec{r}_\varphi \times \vec{r}_\theta = 9\sin\varphi\begin{bmatrix}\sin\varphi\cos\theta\\ \sin\varphi\sin\theta\\ \cos\varphi\end{bmatrix}$$



$$\int_0^{2\pi}\int_0^{\pi}\begin{bmatrix}0\\ 1\\ 3\cos\varphi\end{bmatrix}\cdot\begin{bmatrix}9\sin^2\varphi\cos\theta\\ 9\sin^2\varphi\sin\theta\\ 9\sin\varphi\cos\varphi\end{bmatrix} d\varphi\, d\theta$$

$$\int_0^{2\pi}\int_0^{\pi} 9\sin^2\varphi\sin\theta + 27\sin\varphi\cos^2\varphi \; d\varphi\, d\theta$$

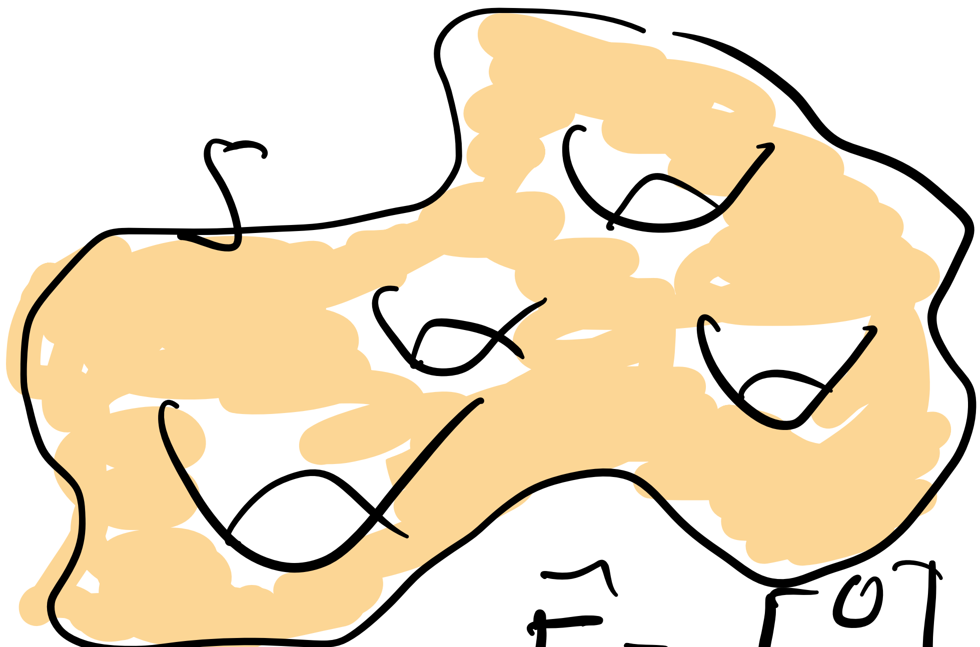
$$\frac{1-\cos 2\varphi}{2}$$

$$\int_0^{2\pi} \frac{9}{2}\sin\theta - 27\frac{\cos^3\varphi}{3}\Big|_0^{\pi} \quad d\theta$$

$$\underbrace{-\frac{9}{2}\cos\theta\Big|_0^{2\pi}}_{0} - 27\left(\frac{\cos^3(\pi)-\cos^3(0)}{3}\right)$$

$$= \int_0^{2\pi}\frac{27\cdot 2}{3}\, d\theta = \boxed{36\pi}$$

The triple integral was easier.



$\vec{F} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$

$\iint_S \vec{F} \cdot d\vec{S}$ is the volume of E

The boundary determine the volume

$$\vec{F} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \quad \operatorname{div} \vec{F} = 1$$

$$\vec{F} = \begin{bmatrix} 0 \\ x \end{bmatrix} \quad \operatorname{curl}(\vec{F}) = 1$$

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3 Remarks

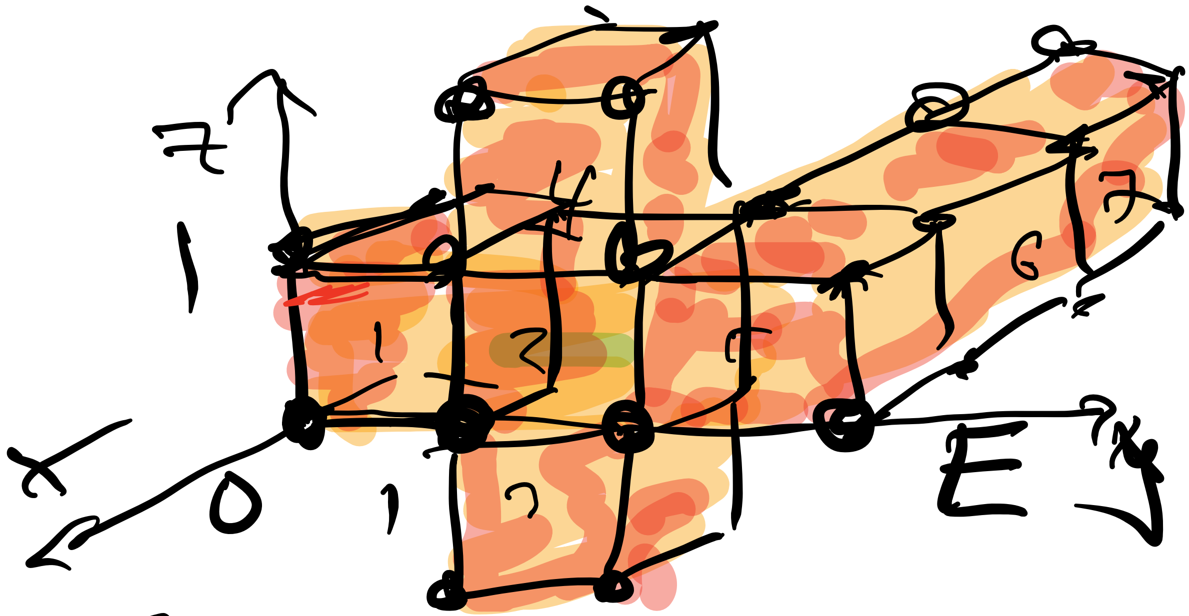
A) What happens if $\vec{F} = \operatorname{curl}(\vec{G})$

$$\iiint_E \operatorname{div} \overset{G}{\operatorname{curl}}(G)\, dV = \iint_S \operatorname{curl}(G) \cdot d\vec{S}$$

why is zero.
The flux of curl($\vec{F}$)
through a closed surface
S is always zero.
We have seen that using
Stokes theorem.

B What happens if
$\mathrm{div}(\vec{F}) = c$
is constant?

Then the flux is
equal to c times
the volumes.



$$\vec{F} = \begin{bmatrix} \sin zy + 3x \\ \sin \sin x \\ e^{yx} + 17z \end{bmatrix}$$

What is the flux of $\vec{F}$ through S, bounding E oriented outwards

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we use the divergence theorem!

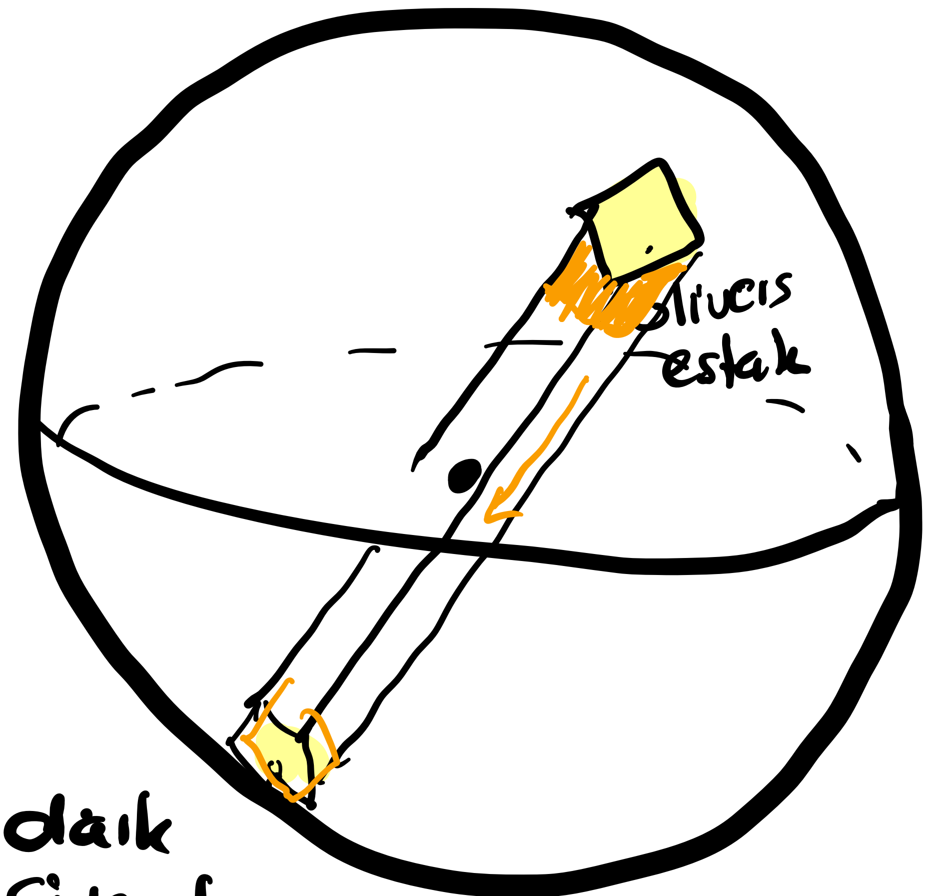
$$\operatorname{div}(\vec{F}) = 20 \qquad 7 \cdot 20 = \boxed{140}$$

$$\iiint \operatorname{div}(\vec{F})\, dV = \iiint 20\, dV$$

E

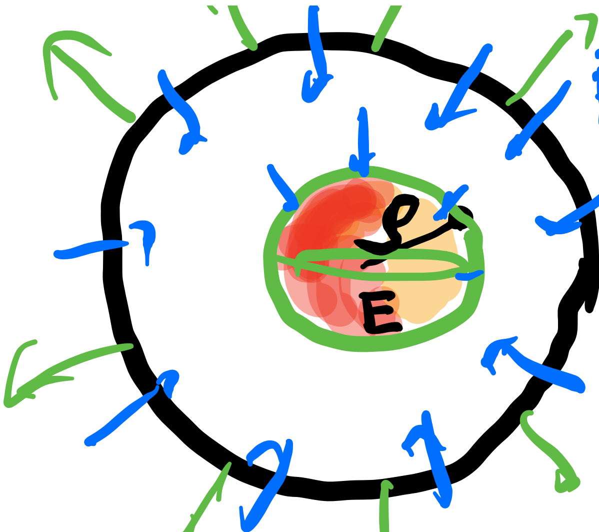
E

(4) Moon plans



dark
Side of
the moon

what is the
gravity at radius
r inside the moon



$$\vec{F} = -F(\rho)\begin{bmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{bmatrix}$$

strength of $\vec{F}$ at distance ρ

Gauss reformulation of gravity as follows:

$$\boxed{\operatorname{div}(\vec{F}) = 4\pi\sigma}$$

mass density

Use the divergence theorem:

$$\iiint_E \operatorname{div}\vec{F}\, dV = \iiint_E 4\pi\sigma\, dV \qquad \sigma = 1$$

$$\| \qquad\qquad = 4\pi \cdot \frac{4\pi}{3}\rho^3 \sigma$$

$$\iint \vec{F}\cdot\vec{dS}$$

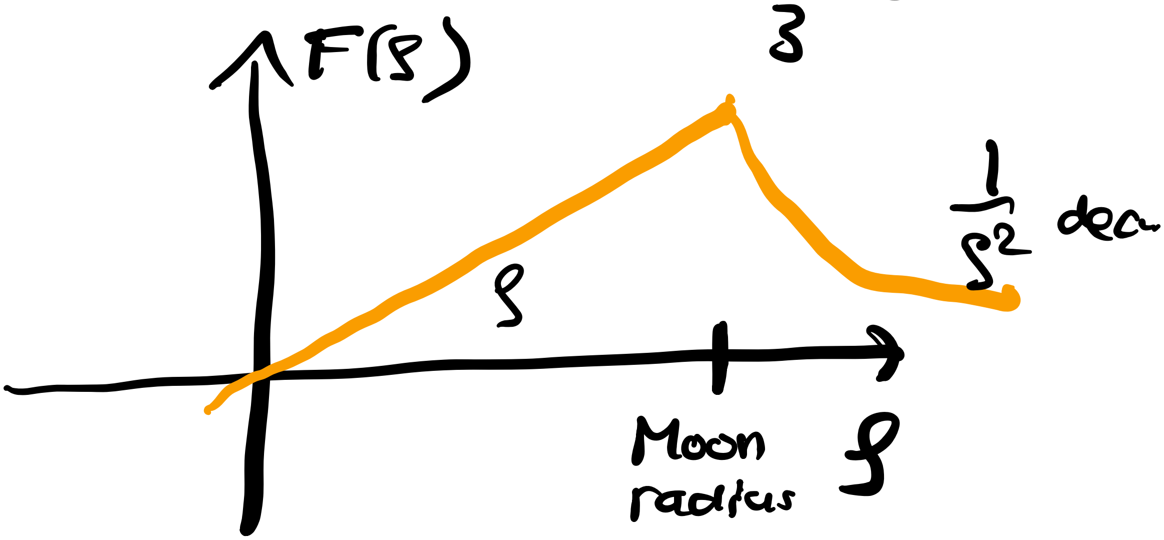
Volume · σ = Mass

S || <u>Homework</u>

$+ F(\rho) \cdot$ surface area

$= + F(\rho) \cdot 4\pi\rho^2$

so : $F(\rho) = \frac{4\pi}{3}\rho\sigma$



what happens for outside the moon.

$$\iiint_E 4\pi\sigma \, dv = M \cdot 4\pi$$

mass of moon

if $\rho > \rho_{\text{radius of moon}}$

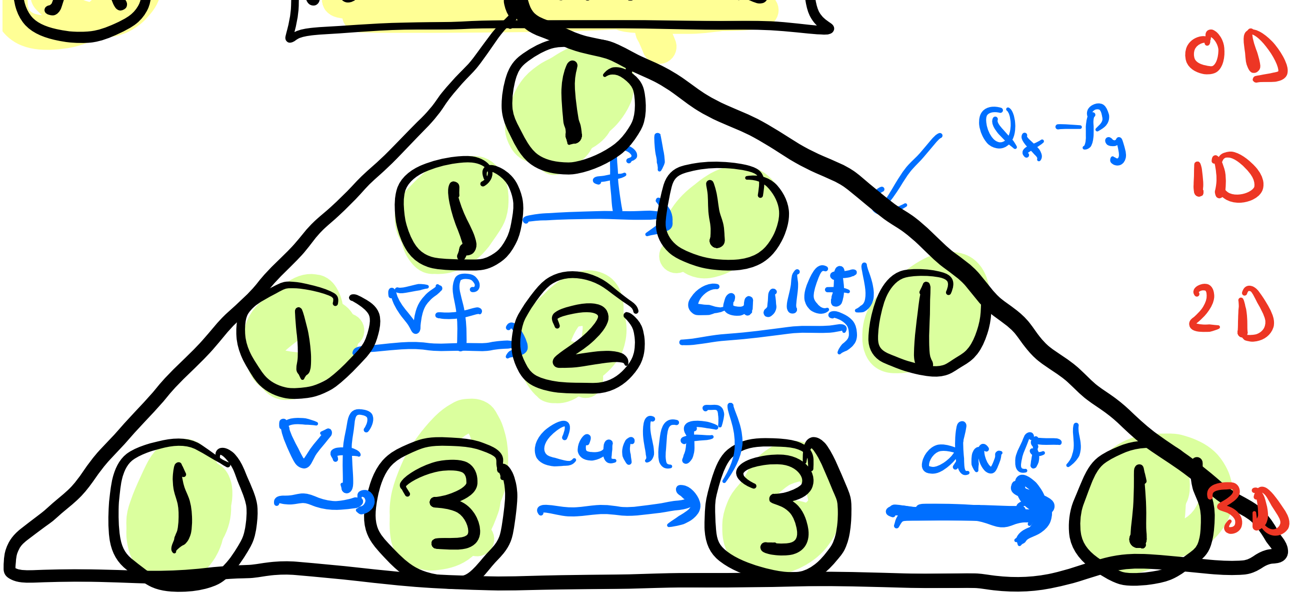
$$F(\rho)\,\cancel{4\pi}\rho^2 = \cancel{4\pi} M$$

$$\boxed{F(\rho) = \frac{M}{\rho^2}}$$

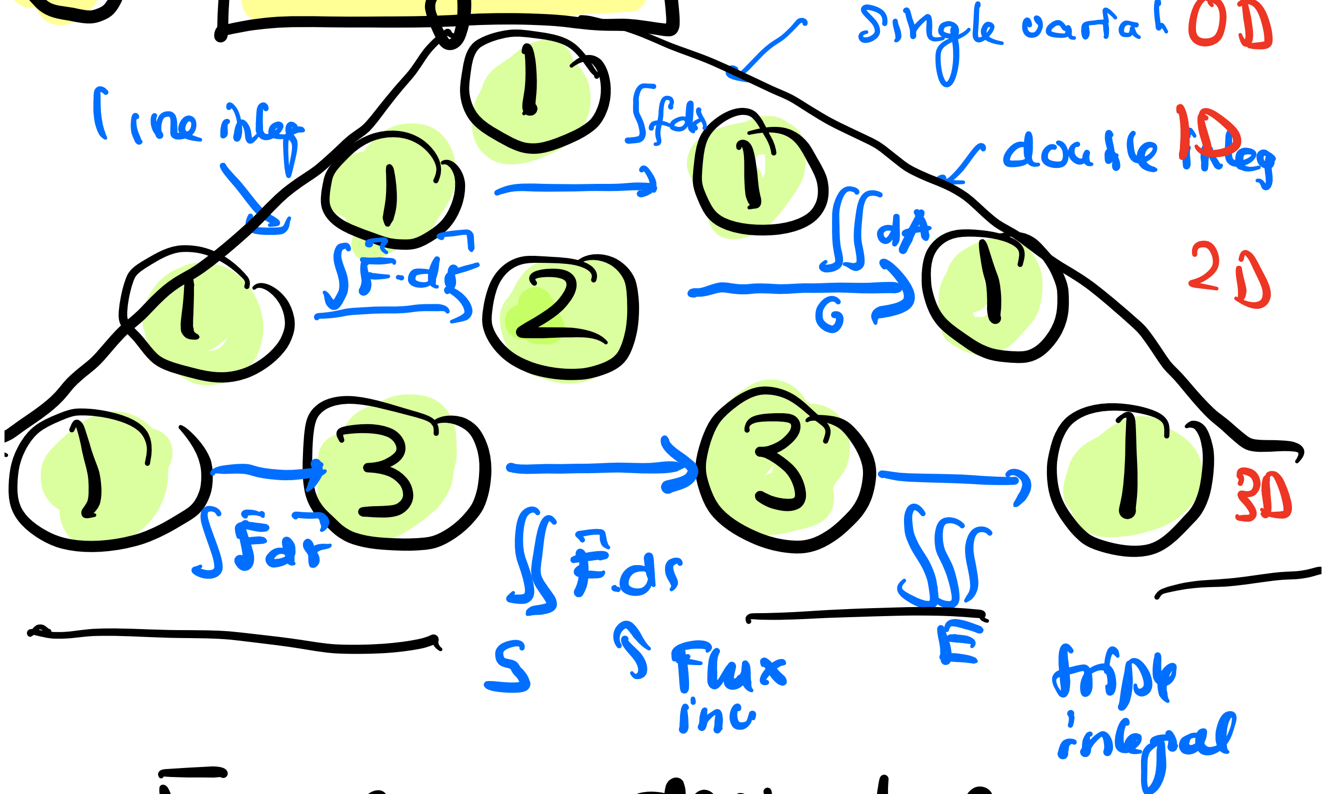
good old Newton's law of gravity.

$\boxed{\text{3 diagrams}}$

(A) Derivatives



(B) Integrals



For every derivative there is an integral and an integral theorem

(C) Theorems

This actually continues to any dimension

$$
\begin{array}{c}
1 \\
1 \to 1 \\
1 \to 2 \to 1 \\
1 \to 3 \to 3 \to 1 \\
1 \to 4 \to 6 \to 4 \to 1
\end{array}
$$

How to continue without cross product? not in this course