7/7/2022 FIRST HOURLY Practice 1 **Maths 21a, O.Knill, Summer 2022**

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
Total:		120

Problem 1) (20 points) No justifications are needed.

1) T **F** For any vector $\vec{v} = [a, b, c]$ we have $|[a, b, c]| = [|a|, |b|, |c|]$.

Solution:

We used the absolute sign for vectors as magnitude.

2) **T** F The curvature of a curve at a point is independent of the parametrization.

Solution:

A basic property.

3) T **F** It is possible to intersect a cylinder with a plane and get a hyperbola.

Solution:

The intersection is either empty, a line or then an ellipse.

4) **T** F If $\vec{T}, \vec{N}, \vec{B}$ is a TNB frame then $\vec{N} = \vec{B} \times \vec{T}$

Solution:

The three vectors are an orthonormal frame like $\vec{T} = [1, 0, 0], \vec{N} = [0, 1, 0], \vec{B} = [0, 0, 1]$.

5) **T** F The intersection between two spheres of radius 1 and 2 is either empty, a point, a circle.

Solution:

Just draw all the possible situations.

6) T **F** The set of points in space which satisfy $x^2 - y^2 = 1$ form a hyperbola.

Solution:

It is a cylindrical hyperboloid.

7) T **F** The length of the sum of two vectors in space is always larger or equal than the sum of the lengths of the vectors.

Solution:

Just opposite

8) **T** F For any three vectors $\vec{u}, \vec{v}, \vec{w}$, the identity $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + \vec{u} \times \vec{w}$ holds.

Solution:

One calls this distributivity.

9) **T** F The set of points which satisfy $-x^2 - 2x + y^2 + z^2 = 1$ define a cone.

Solution:

Complete the square.

10) **T** F If A, B, C are three points space which are not contained in a common line, then $\vec{AB} \times \vec{AC}$ is a vector orthogonal to the plane containing A, B, C.

Solution:

The condition assures that the vectors $\vec{AB}$ and $\vec{AC}$ are not parallel.

11) **T** F The line $\vec{r}(t) = [t, 5t, 4t]$ hits the plane $x + 5y + 4z = 100$ at a right angle.

Solution:

Indeed, the normal vector $[1, 5, 4]$ is also the velocity vector of the curve.

12) T **F** The surface given in (r, θ, z) coordinates as $r = \sin(\theta)$ is a paraboloid.

Solution:

It translates to $r^2 = r\sin(\theta)$ which is $x^2 + y^2 = y$ which is actually a cylinder.

13) **T** F If $\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$, then $\vec{v}$ and $\vec{w}$ are parallel (in the sense that there exists a constant c such that $\vec{v} = c\vec{w}$).

Solution:

The condition means that the cross product is zero implying parallel vectors.

14) **T** F If $|\vec{x} \times \vec{v}| = 0$ for all vectors $\vec{v}$, then $\vec{x} = \vec{0}$.

Solution:

That means that $\vec{x}$ is perpendicular to all vectors. This is only possible for the zero vector.

15) **T** F If $\vec{u}$ and $\vec{v}$ are orthogonal, then $(\vec{u} \times \vec{v}) \times \vec{u}$ is parallel to $\vec{v}$.

Solution:

$\vec{u}, \vec{v}, \vec{w} = \vec{u} \times \vec{v}$ are all perpendicular. The vector under consideration is perpendicular both to $\vec{w}$ and to $\vec{u}$ and so parallel to $\vec{v}$.

16) T **F** Every vector contained in the line $\vec{r}(t) = [4 + 2t, 2 + 3t, 3 + 4t]$ is parallel to the vector $(4, 2, 3)$.

Solution:

It is parallel to the vector $[2, 3, 4]$.

17) **T** F If in spherical coordinates a point is given by $(\rho, \theta, \phi) = (1/2, 3\pi/2, \pi/2)$, then its rectangular coordinates are $(x, y, z) = (0, -1/2, 0)$.

Solution:

The angle conditions means to be on the negative y-axes. Now take distance $1/2$.

18) T F The set of points which satisfy $x^2 - 4x + 2y^2 + 3z^2 = -3$ is an ellipsoid.

Solution:

Complete the square again.

19) T F If $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$, then all three vectors $\vec{u}, \vec{v}, \vec{w}$ are in the same plane.

Solution:

Indeed the volume of the parallel epiped is zero.

20) T F The set of points in $\mathbb{R}^3$ which have distance 1 from the curve $\vec{r}(t) = [3\cos(t), 3\sin(t), 0]$ form a torus (doughnut)

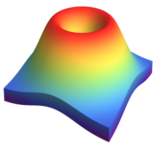
Solution:

Yes, that is a way how you can draw a doughnut. Take a circle away from the z axes and spin it around the z axes.

Problem 2) (10 points) No justifications are needed in this problem.

In each sub-problem, each of the numbers 0,1,2,3 each occur exactly once.

a) (2 points) Match the graphs of the functions $f(x,y)$. Enter 0 if there is no match.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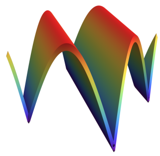
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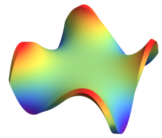
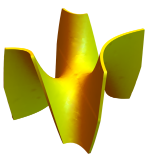

3

Function $f(x,y) =$	0,1,2, or 3
$xy(x^2 - y^2)$	
$e^{-x^2-y^2}(x^2 + y^2)$	
$1/(x^2 + y^4 + 1)$	
$\|\sin(x + y)\|$	

b) (2 points) Match the surfaces $g(x,y,z) = c$. Enter 0 if there is no match.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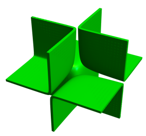
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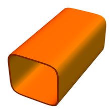
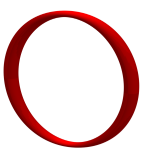

3

Function $g(x,y,z) =$	0,1,2, or 3
$1000 * \sin(xyz) = 1$	
$x^6 + z^6 = 1$	
$z - (x^2 - y^2)x = 0$	
$x - y^2 = 1$	

c) (2 points) Match the space curves with the parametrizations. Enter 0 if there is no match.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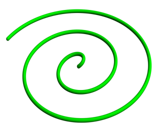
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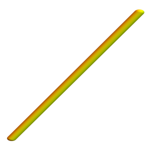
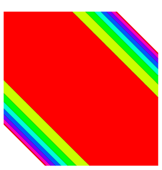

3

Parametrization $\vec{r}(t) =$	0,1,2, or 3
$[t\cos(t), t\sin(t), \sin(t)]$	
$[\cos(3t), 0, 3\sin(3t)]$	
$[\exp(t), 2\exp(t), 3\exp(t)]$	
$[t, 0, t\sin(t)]$	

d) (2 points) Match the functions g with contour plots in the xy-plane. Enter 0 if there is no match.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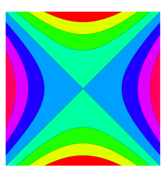
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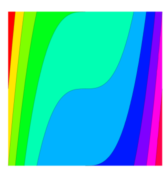
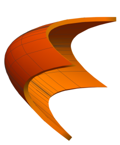

3

Function $g(x,y) =$	0,1,2, or 3
$x^2 - y^2$	
$(x + y)^4$	
$x^3 - y$	
$\cos(3x) + \sin(3y)$	

e) (2 points) Match the parametrized surfaces. Enter 0 if there is no match.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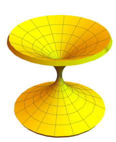
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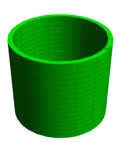
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Parametrization $\vec{r}(u,v) =$	0-3
$[\cos(v), \sin(v), u]$	
$[u^2\cos(v), u^2\sin(v), u]$	
$[\sin(v)\cos(u), \cos(v), 2\sin(v)\sin(u)]$	
$[u^6 + v^6, u^3, v^3]$	

Solution:

a) 3102
b) 2310
c) 2130
d) 2130
e) 3201

Problem 3) (10 points)

We perform some computations with the vectors $\vec{v} = [2, 2, 3]$ and $\vec{w} = [1, 2, 2]$.

a) (2 points) Find the cross product $\vec{v} \times \vec{w}$?

b) (2 points) Construct a unit vector in the same direction than $\vec{v} \times \vec{w}$.

c) (2 points) Find $\cos(\alpha)$ for the angle α between $\vec{v}, \vec{w}$.

d) (2 points) What is the vector projection $\vec{P}_{\vec{w}}(\vec{v})$ of $\vec{v}$ onto $\vec{w}$?

e) (2 points) Compute $(\vec{v} + \vec{w}) \cdot (\vec{v} \times \vec{w})$.

Solution:

a) $[-2, -1, 2]$
b) $[-2, -1, 2]/3$.
c) $12/(3\sqrt{17})$.
d) $12/9[1, 2, 2]$.
e) 0.

Problem 4) (10 points)

a) (2 points) Parametrize the plane containing the points $(1, 0, 0), (0, 5, 0), (0, 0, 2)$ using parameters s, t:

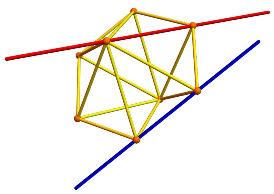
b) (3 points) Now find the equation $ax + by + cz = d$ of that plane in a).
c) (5 points) Finally find the distance between that plane defined in a) and the point $P = (3, 3, 3)$.

Solution:

a) A possibility is $\vec{r}(s, t) = [1 - t - s, 5t, 2s]$.
b) $10x + 2y + 4z = 10$.
c) $n = [10, 2, 4]$ is a normal vector. This means the distance is $[3 - 1, 3, 3].[10, 2, 4]/|[10, 2, 4]| = 38/\sqrt{120}$.

Problem 5) (10 points)

A **prismatic polyedron** with 8 vertices contains the 4 points $A = (2, 0, 0), B = (1, \sqrt{3}, 0)$ and $C = (-1, \sqrt{3}, 0), D = (0, 0, \sqrt{3})$. Find the distance of the line containing the points A, B and the line containing the points C, D.



Solution:

This is a standard distance line-line problem: we have $\vec{AB} = [-1, \sqrt{3}, 0]$, $\vec{CD} = [1, -\sqrt{3}, \sqrt{3}]$, $\vec{BC} = [2, 0, 0]$. We have $\vec{AB} \times \vec{CD} = [3, Sqrt[3], 0]$ with $|\vec{AB} \times \vec{CD}| = \sqrt{12}$ The distance is $d = \vec{BC} \cdot (\vec{AB} \times \vec{CD})/|\vec{AB} \times \vec{CD}| = 6/\sqrt{12}$. This simplifies to $\sqrt{3}$.

Problem 6) (10 points)

a) (5 points) Find the arc length of the path

$$\vec{r}(t) = [\frac{t^3}{3}, t^2, 2t]$$

with $-1 \leq t \leq 1$.

b) (5 points) Compute $\vec{v} = \vec{r}\,'(1), \vec{w} = \vec{r}\,''(1)$ and express the curvature of the curve at $\vec{r}(1)$ in terms of $\vec{v}$ and $\vec{w}$.

Solution:

a) We have $\vec{r}\ '(t) = [t^2, 2t, 2]$. The speed is $|\vec{r}\ '(t) = 2 + t^2$. The integral is 14/3.
b) $\vec{v} = [1, 2, 2]$
$\vec{w} = [2, 2, 0]$
$\kappa = |\vec{v} \times \vec{w}|/|vecv|^3 = |[-4, 4, -2]|/3^3 = 6/27 = 2/9$.

Problem 7) (10 points)

On Mars, the sum of gravitational acceleration and wind force is $\vec{r}\,''(t) = [0, \sin(t), -4]$. The **Mars helicopter Ingenuity** has been rising up to 5 meters. There had been a small stone pebble stuck on one of the legs. It falls down from the position $\vec{r}(0) = [2, 1, 5]$ with velocity $\vec{r}\,'(0) = [1, 0, 0]$ subject to the acceleration given above.

a) (6 points) Determine the path of the pebble.

b) (4 points) At which time does the pebble hit the ground?

Solution:

a) Integrate and fix the constant to get $\vec{r}\,'(t) = [1, 1 - \cos(t), -4t]$.
Integrate and again and fix the constant $\vec{r}(t) = [t + 2, t - \sin(t) + 1, 5 - 2t^2]$.
b) We need $5 - 2t^2 = 0$ means $t = \sqrt{5/2}$.

Problem 8) (10 points)

We denote with $|ABC|$ the **area** of a triangle ABC defined by three points A, B, C. Take $A = (2, 0, 0), B = (0, 3, 0), C = (0, 0, 1)$ as well the point $O = (0, 0, 0)$. Verify in this case the **3D Pythagoras theorem**

$$|ABC|^2 = |ABO|^2 + |BCO|^2 + |ACO|^2 .$$

$|ABC|^2 =$

$\|ABO\|^2 =$	$\|BCO\|^2 =$	$\|ACO\|^2 =$

Solution:

We have $|ABC| = |\vec{AB} \times \vec{AC}|/2 = 7/2$. This could not be short cut. We also have $|ABO| = |\vec{OA} \times OB| = 3$. etc. (The three triangle areas square $|ABO|^2 = 3^2$, $|BCO|^2 = 9/4$ and $|ACO|^2 = 1$ could also be computed directly.)

Problem 9) (10 points) No justifications are needed.

a) (2 points) Parametrize the surface $3x + 2y + 4z = 12$.

$\vec{r}(s,t) = [\quad , \quad , \quad]$

b) (2 points) Parametrize the surface $y^2 + (z-1)^2 = 9$ using an angle θ in the yz-plane.

$\vec{r}(\theta, x) = [\quad , \quad , \quad]$

c) (2 points) Parametrize the surface $x^2 + 2x + y^2 + z^2/9 = 0$.

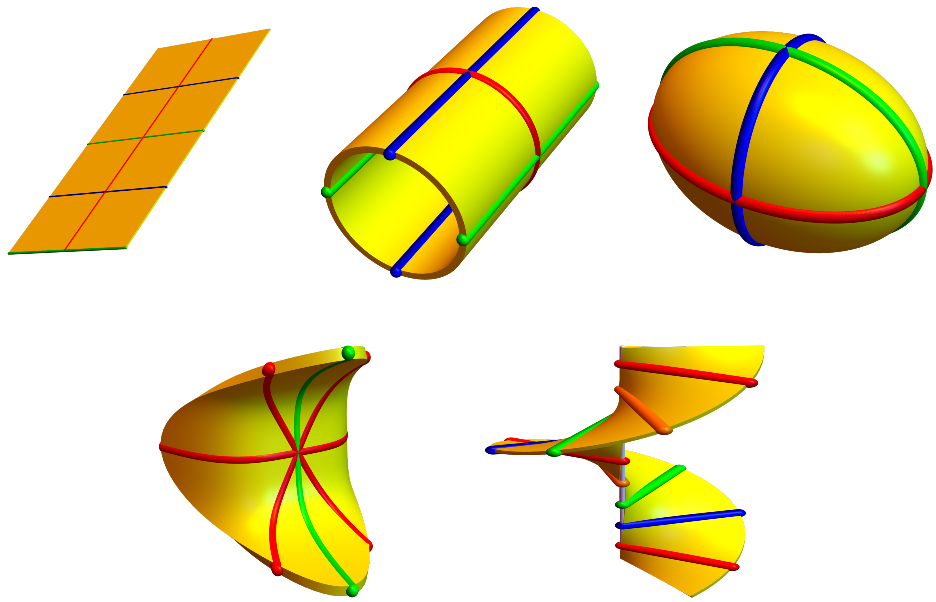
$\vec{r}(\theta, \phi) = [\quad , \quad , \quad]$

d) (2 points) Parametrize the surface $y = x^4 - z^4$.

$\vec{r}(x, z) = [\quad , \quad , \quad]$

e) (2 points) Parametrize the surface obtained by taking the helix $r(\vec{t}) = [\cos(t), \sin(t), t]$ and connect each point $\vec{r}(t)$ with the projection onto the z-axes.

$\vec{r}(t, s) = [\quad , \quad , \quad]$



Solution:

Example solutions: a) $[s, t, (12 - 3s - 2t)/4]$
b) $[x, 3\cos(t), 1 + 3\sin(t)]$
c) $[\sin(\phi)\cos(\theta) - 1, \sin(\phi)\sin(\theta), 3\cos(\phi)]$
d) $[x, x^4 - z^4, z]$
e) $[s\cos(t), s\sin(t), t]$