7/7/2022 FIRST HOURLY Practice 3 **Maths 21a, O.Knill, Summer 2022**

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
Total:		120

Problem 1) (20 points) No justifications are needed.

1) T F The plane $3x + 4y + z = 4$ intersects the y-axes in the point $(0, 1, 0)$.

2) T F The Cauchy-Schwartz inequality implies that for two unit vectors $\vec{v}$ and $\vec{w}$ the dot product $\vec{v} \cdot \vec{w}$ is in the interval $[0, 1]$.

3) T F Assume we have three points A, B, C such that $|\vec{AB}| = |\vec{AC}|$ then $A = C$.

4) T F The curve $\vec{r}(t) = [t^3, 2 + 2t^3, 3 + 3t^3]$ is a line.

5) T F If a smooth curve has curvature 1 everywhere, then its speed $|\vec{r}\,'(t)|$ is constant 1 everywhere.

6) T F The surface $x^2 - y + y^2 = z$ is a hyperbolic paraboloid.

7) T F The angle between two vectors is always a number in $[0, \pi/2]$.

8) T F If the velocity of a curve $\vec{r}(t)$ is a constant vector $\vec{v}$, then the curve is a line.

9) T F The lines $\vec{r}(t) = [0, t, -t]$ and $\vec{r}(t) = [t(1 - t), t, 1 - 2t]$ do intersect.

10) T F The formula given in spherical coordinates as $\rho = \rho \cos^2(\phi)$ defines a union of two planes.

11) T F If $|\vec{u} \times \vec{v}| = 0$, then $\vec{u}$ and $\vec{v}$ are parallel in the sense that there exists a real number λ for which $\vec{u} = \lambda\vec{v}$.

12) T F The curve given in polar coordinates as $\sin(\theta) + \cos(\theta) = r$ is a circle.

13) T F The arc length a curve $\vec{r}(t)$ with $0 \leq t \leq 1$ is $|\int_0^1 \vec{r}\,'(t)\, dt|$.

14) T F The surface parametrized by $\vec{r}(\phi, \theta) = [\phi, \phi^2 + \theta^2, \theta]$ is an elliptic paraboloid.

15) T F There is a time t, when the velocity vector of $\vec{r}(t) = [\cos(t), \sin(t), t]$ is parallel to the vector $[0, 0, 1]^t$.

16) T F It is possible that the intersection of two ellipsoids is a hyperbola.

17) T F The function $f(x, y) = \sqrt{x^4 + y^3 + 1}$ has the entire plane as its domain.

18) T F The bi-normal vector $t \to \vec{B}(t)$ is a vector which always has a positive z component.

19) T F The distance between two parallel planes is the distance of a point P in one plane to the other plane.

20) T F Given three vectors $\vec{u}, \vec{v}, \vec{w}$, then the vectors $\vec{u} + \vec{v}$, $\vec{u} + \vec{w}$, $\vec{v} - \vec{w}$ always are contained in a plane.

Total

Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the contour surfaces $g(x, y, z) = 0$. Enter O, if there is no match.

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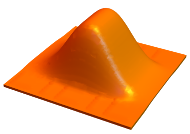

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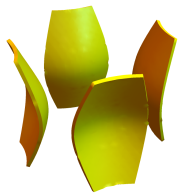

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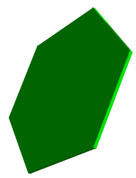

Function $g(x, y, z) =$	O,I,II or III
$x^2y^2 - z^2 = 1$	
$y^2 - z^8 = 1$	
$x + y + z = 0$	
$z - 2\exp(-x^2 - y^6) = 0$	

b) (2 points) Match the graphs of the functions $f(x, y)$. Enter O, if there is no match.

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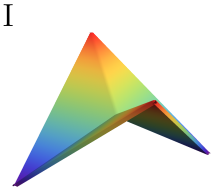

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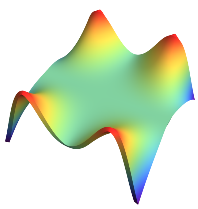

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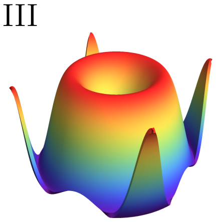

Function $f(x, y) =$	O,I,II or III
$\sin(x^2 + y^2)$	
$\lvert x - y\rvert - \lvert x + y\rvert$	
$y^2 \sin(x^2)$	
$\sin(x)$	

c) (2 points) Match the space curves with the parametrizations. Enter O, if there is no match.

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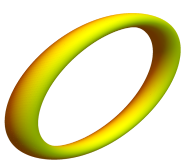

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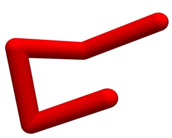

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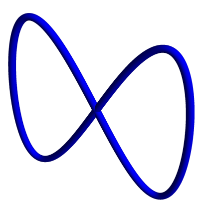

Parametrization $\vec{r}(t) =$	O, I,II or III
$[\cos(t), 0, \sin(2t)]$	
$[\lvert t\rvert, \lvert t - 1\rvert, \lvert\lvert t\rvert - 1\rvert]$	
$[0, \sin(2t), \cos(2t)]$	
$[t\cos(t), 0, t\sin(t)]$	

d) (2 points) Match the functions g with contour plots in the xy-plane. Enter O, if there is no match.

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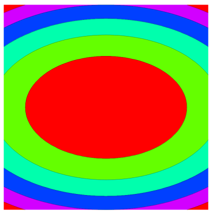

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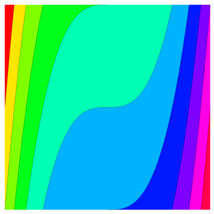

III

Function $g(x, y) =$	O, I,II or III
$2x^2 + 5y^2$	
$x - y$	
$x^3 - y$	
$\sin(x) + \sin(y)$	

e) (2 points) Match the surfaces. Enter O if there is no match.

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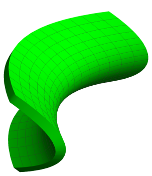

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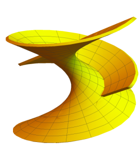

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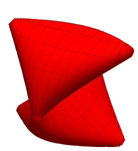


Surface	O - III
$[u^2 - v^2, 2u, v]$	
$[u\cos[v], u\sin[v], v + u]$	
$[\cos(u)\cos(v), \cos(u)\sin(v), \sin(u)]$	
$[u^2, u^2 - v^2, v^2]$	

Problem 3) (10 points)

Similarly as a pianist must practice etudes, or an athlete needs to push weights, a mathematician must practice basic computations. Our theme is build from the vectors

$$\vec{v} = \begin{bmatrix} 3 \\ 4 \\ 12 \end{bmatrix}, \ \vec{w} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} .$$

Please play as Allegro Sostenuto.

a) (2 points) What is the length of $\vec{v}$ and of $\vec{w}$?

b) (2 points) What is the dot product $\vec{v} \cdot \vec{w}$?

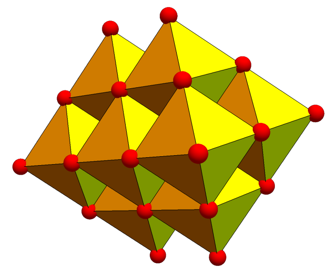
c) (2 points) What is the cross product $\vec{v} \times \vec{w}$?

d) (2 points) Find $\cos(\alpha)$ for the angle α between $\vec{v}, \vec{w}$.

e) (2 points) What is the projection $\vec{P}_{\vec{w}}(\vec{v})$ of $\vec{v}$ onto $\vec{w}$?

Problem 4) (10 points)

Molybdates are compounds containing molybdenum and oxygen. An example is **decacoltanate** $M_{10}O_{28}$. Molybdates belong to the larger class of Polyoxometalates (POM) and are a hot spot in chemistry due to many applications, like pigments, batteries, semiconductors, photoactive materials etc. A decacoltanate can be visulized by 10 octahedra where each contains a Molybdenum atom. Each of the 10 octahedra is made of 8 triangles so that there are 80 triangles in this structure.



a) (3 points) Assume that one of the triangles has the vertex coordinates

$$A = (1, 2, 0), B = (0, 1, 2), C = (2, 0, 1) .$$

Find the area of the triangle ABC. Then find the total surface area of the structure.

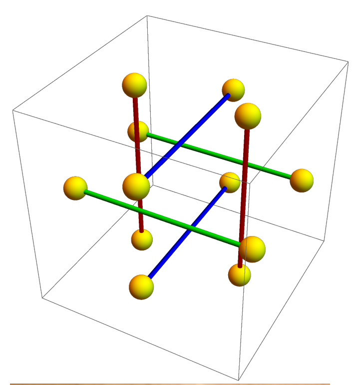
b) (3 points) What is the equation $ax + by + cz = d$ for the plane through A, B, C?

c) (2 points) Parametrize the line $\vec{r}(t)$ passing through A which is perpendicular to the triangle.

d) (2 points) What is the cos of the angle between the two vectors $\vec{AB}$ and $\vec{AC}$?

Problem 5) (10 points)

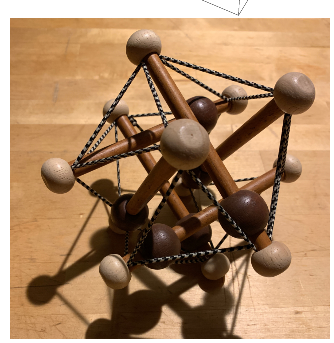
Oliver recently got an **icosahedron tensegrity model** that is made of 6 strouts. **Tensegrity** stands for tension and integrity and is a structural principle in architecture. Use a distance formula to get the distance between the line connecting A and B with



$$A = (-\phi, 1, 0), B = (\phi, 1, 0)$$

and the line connecting

$$C = (0, -\phi, 1), D = (0, \phi, 1) .$$



Here, $\phi = (1 + \sqrt{5})/2$ is the **golden ratio**. You can work with the letter ϕ in your computation and leave the result in terms of ϕ if you like.

Tensegrity is also used as a meditation mantra. Just repeat the sentence

"I remain stable even so I'm stressed and tense"

again and again and everything is fine.

Problem 6) (10 points)

During the 4th of July celebration near the **Charles river basin**, a rocket was observed to fly on the curve

$$\vec{r}(t) = \left[\frac{t^6}{6}, \frac{\sqrt{2}t^5}{5}, \frac{t^4}{4} \right] .$$

a) (2 points) Find the velocity of $\vec{r}(t)$ and the unit tangent vector.

b) (2 points) Is the unit tangent vector $\vec{T}$ defined at $t = 0$, the time the rocket lifts off? If yes, what is it, if no, why not?

c) (6 points) What is the arc length of the curve parametrized with $t \in [0, 1]$?

Problem 7) (10 points)

The US soccer team won the world cup final. A memorable moment during the tournament was the tea drinking quip of **Alex Morgan** after scoring a goal against England in the semi-final. Alex hit the ball with a head shot. Assume the ball was hit at the position and velocity

$$\vec{r}(0) = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}, \vec{r}\,'(0) = \begin{bmatrix} 10 \\ 5 \\ 2 \end{bmatrix} .$$

and was subject to the force (acceleration)

$$\vec{r}\,''(t) = \begin{bmatrix} 0 \\ 1 \\ -10 \end{bmatrix} .$$

When and where does the ball hit the goal which is part of the surface $y = 21$?

Problem 8) (10 points)

a) (4 points) The notation $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is due to Hamilton. Lets parse some expressions. Note that also in multivariable calculus, there is the PEMDAS rule, first parenthesis, exponentials, multiplication and divisions, then addition and subtraction.

The object	is vector	is scalar	is not defined
$(\vec{i}+1)\times(\vec{j}+1)$			
$(\vec{i}+\vec{k})\times(\vec{j}+\vec{k})$			
$\vec{i}+(\vec{i}\times\vec{j})+\vec{j}$			
$1+(\vec{i}\cdot\vec{j}+\vec{k})+1$			

By the way, there is a lot of more room for arithmetic here. You can ask yourself for example, what is $\vec{i}^{\vec{i}}$. There is mathematics which can make sense of this and assign a value like $e^{-\pi/2}$. But that needs the world of quaternions.

b) (4 points) Which expressions are independent of the curve parametrization?

Expression	parameter independent	parameter dependent
curvature of a curve at a point		
the arc length of a curve		
the velocity of a curve		
the unit tangent vector of a curve		

c) (1 point)

Write down the Cauchy-Schwarz inequality! Remember that we had proven that on the first day.

d) (1 point)

Write down a possible formula for the curvature of a curve $\vec{r}(t)$.

Problem 9) (10 points) No justifications are needed.

We have just had a few great hot summer days. While working for the exam, we enjoyed a **cool iced macchiato** with cream and a cherry on top. There are various ingredients. The cup is part of a cone, the bottom is a disc, the straw is a cylinder, the cream is a tube, the lemon slice is a graph of a function. Please complete the parametrizations. We don't ask you to parametrize the cream. It will be up to you to think about the next time you enjoy a nice caffè mocha.

a) (2 points) The **cup surface** is $x^2 + y^2 = z^2/25$.

$\vec{r}(r,\theta) = [\quad , \quad , \quad]$

b) (2 points) The **waffle lemon slice** $x = 4 - \sin(yz/5)$.

$\vec{r}(y,z) = [\quad , \quad , \quad]$

c) (2 points) The **cherry** $x^2 + y^2 + (z-14)^2 = 1$.

$\vec{r}(\theta,\phi) = [\quad , \quad , \quad]$.

d) (2 points) The **cookie straw** $(x-5)^2 + (z-16)^2 = 1$

$\vec{r}(y,\theta) = [\quad , \quad , \quad]$.

e) (2 points) The **table plane** $z = 15$ containing the **coaster** $z = 15$.

$\vec{r}(x,y) = [\quad , \quad , \quad]$