7/6/2023 FIRST HOURLY Practice 1 **Maths 21a, O.Knill, Summer 2023**

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) T F A circle of radius 20 has curvature 20.

Solution:

It has curvature 1/20.

2) T F The identity $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|$ implies that $\vec{v}$ and $\vec{w}$ are parallel.

Solution:

Indeed, this implies $\cos(\alpha) = 1$ which means $\alpha = 0$.

3) T F The cross product is associative so that $[0,0,1] \times ([0,1,0] \times [0,1,0]) = ([0,0,1] \times [0,1,0]) \times [0,1,0]$.

Solution:

This is a counter example to associativity.

4) T F The vector $\vec{v} = [1,1,1]$ containing only units is a unit vector.

Solution:

It is a vector of length $\sqrt{3}$.

5) T F The function $f(x,y) = x^3y^2 + 1$ has a root.

Solution:

Take $y = 1$ to get a function of one variable $g(x) = x^3 + 1$. This function is positive for $x = 2$ and negative for $x = -2$ so that there.

6) | T | **F** | There are two vectors for which the angle between the two vectors is $-\pi/2$.

Solution:

The angle between two vectors is defined to be a vector in the interval $[0, \pi]$.

7) | T | **F** | We have $|\vec{v} \times \vec{v}| = |\vec{v}|^2$ for all vectors $\vec{v}$.

Solution:

This holds for the dot product, not the cross product.

8) | **T** | F | The curvature of the curve $r(t) = [2\cos(t^3), 2\sin(t^3)]$ is constant $1/2$.

Solution:

The curve is a circle of radius 2. The curvature is $1/2$.

9) | T | **F** | It is possible to intersect a hyperbolic paraboloid with a plane and get an ellipse.

Solution:

The intersection is always a line, a parabola or a hyperbola going to infinity.

10) | **T** | F | If $\vec{T}, \vec{N}, \vec{B}$ is a TNB frame, then $\vec{B} \cdot (\vec{T} \times \vec{N}) = 1$.

Solution:

Indeed $\vec{B} = (\vec{T} \times \vec{N})$ and since it has length 1, $\vec{B} \cdot \vec{B} = 1$.

11) | **T** | F | The set of points in space which satisfy $x^2 - 2x - y^2 - 2y + z^2 - 2z = 1$ is a one-sheeted hyperboloid.

Solution:

Yes it is

12) T **F** The length of the cross product of $\vec{v}$ and $\vec{w}$ can be larger than $|\vec{v}||\vec{w}|$.

Solution:

We have seen $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\alpha)$.

13) **T** F If $\vec{v}(t), \vec{w}(t)$ are curves, then the derivative $(\vec{v} \cdot \vec{w})'$ is $\vec{v}\,' \cdot \vec{w} + \vec{v} \cdot \vec{w}\,'$.

Solution:

This is the product rule we have used for example to check that $\vec{N}$ and $\vec{T}$ are perpendicular

14) T **F** The set of points given in spherical coordinates as $\rho^2 \sin^2(\phi) - \rho^2 \cos^2(\phi) = 1$ is a two sheeted hyperboloid.

Solution:

It is one sheeted.

15) **T** F If A, B, C are three points space lying on a line, then $\vec{AB} \times \vec{AC}$ is the zero vector.

Solution:

Indeed, the area of the parallel epiped is then zero.

16) T **F** The line $\vec{r}(t) = [3t, 4t, 0]$ hits the plane $-4x + 3y = 10$ at a right angle.

Solution:

The line is actually parallel to the plane

17) T **F** The surface given in (r, θ, z) coordinates as $r = \cos(\theta)$ is an elliptic paraboloid.

Solution:

It translates to $r^2 = r\cos(\theta)$ which is $r^2 = x$ or $x^2 + y^2 = x$.

18) **T** F If in spherical coordinates a point is given by $(\rho, \theta, \phi) = (1, 3\pi/2, \pi/2)$, then its rectangular coordinates are $(x, y, z) = (0, -1, 0)$.

Solution:

It has to be on the y axes.

19) **T** F The point $(1, -1)$ in $\mathbb{R}^2$ has the polar coordinates $(r, \theta) = (\sqrt{2}, 7\pi/4)$.

Solution:

Just check $(\sqrt{2}\cos(-\pi/4), \sqrt{2}\sin(-\pi/4)) = (1, -1)$.

20) **T** F The surface given in spherical coordinates as $\rho \sin^2(\phi) = \cos(\phi)$ is a paraboloid.

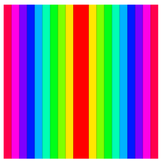
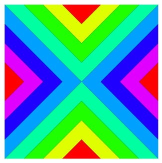
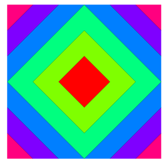
Solution:

Indeed, after multiplying both sides with ρ we see $r^2 = z$.

Problem 2) (10 points) No justifications are needed in this problem.

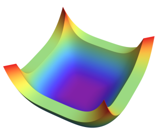
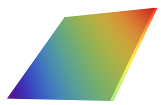
In every sub-problem, each of the numbers 0,1,2,3 each occurs exactly once.

a) (2 points) Match functions g with their xy-contour plots. Enter 0 if there is no match.

1  2  3 

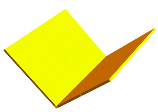
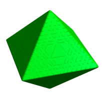
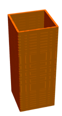
Function $g(x, y) =$	0,1,2, or 3
$\lvert x\rvert - \lvert y\rvert$	
$\lvert x\rvert$	
$\lvert y\rvert$	
$\lvert x\rvert + \lvert y\rvert$	

b) (2 points) Match the graphs of the functions $f(x, y)$. Enter 0 if there is no match.

1  2  3 

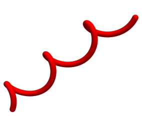
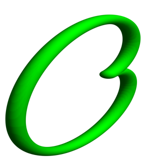
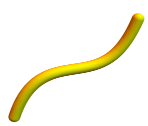
Function $f(x, y) =$	0,1,2, or 3
$x + y$	
$\lvert x + y\rvert$	
x^2y	
$x^4 + y^4$	

c) (2 points) Match the surfaces $g(x, y, z) = c$. Enter 0 if there is no match.

1  2  3 

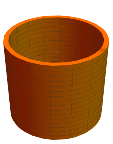
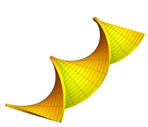
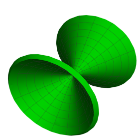
Function $g(x, y, z) =$	0,1,2, or 3
$\lvert x\rvert + \lvert y\rvert + \lvert z\rvert = 1$	
$\lvert x\rvert + \lvert y\rvert = 1$	
$z - \lvert x\rvert = 0$	
$\lvert x\rvert - \lvert y\rvert = 1$	

d) (2 points) Match the space curves with the parametrizations. Enter 0 if there is no match.

1  2  3 

Parametrization $\vec{r}(t) =$	0,1,2, or 3
$[\cos(t), t, \sin(t)]$	
$[t + \cos(3t), 1/t, t + \sin(3t)]$	
$[\sin(t), t, t^3]$	
$[0, \sin(t/2)\cos(t), \sin(t/2)\sin(t)]$	

e) (2 points) Match the parametrized surfaces. Enter 0 if there is no match.

1  2  3 

Parametrization $\vec{r}(u, v) =$	0-3
$[u\cos(v), u, u\sin(v)]$	
$[u\cos(v), v, u\sin(v)]$	
$[\cos(v), \sin(v)\cos(u), 2\sin(v)\sin(u)]$	
$[\cos(v), \sin(v), u]$	

Solution:

a) 2103, b) 3201 c) 2310 d) 1032 e) 3201

Problem 3) (10 points)

We asked 6 Disney figures to give us their most favorite multi-variable algebra calculation problem. Solve their problems! Each is worth 2 points.

a) **Minnie** would love you to compute $([3,4,12] \cdot [12,3,4])^2 + |[3,4,12] \times [12,3,4]|^2$

Answer:

b) **Mickey** wants to know $([1,1,1] \times [1,-1,1]) \cdot [1,1,-1]$

Answer:

c) **Goofy** wants to goof with $([1,0,0] \times [1,0,0]) + ([1,0,0] \cdot [1,0,0])$

Answer:

d) **Donald** demands to know $([1,2,3] \times [3,2,1]) \cdot [1,0,0]$

Answer:

e) **Daisy** wonders $|[1,1,1]|^{|[0,1,0]|}/|[1,1,1]|$

Answer:

f) **Pluto** gets a kick out of $|[1,0,0] + [0,1,0] - [0,0,1]|^2$

Answer:

Oliver's Disney figures during a meeting on what a new music video should be like.

Solution:

a) Using the Cauchy Binet identity this can be done in your head: 13^4
b) 4
c) not defined. Goofy goofed!!!!
d) -4
e) 1
f) 3

Problem 4) (10 points)

a) (3 points) Find the equation $ax + by + cz = d$ of the plane with normal vector $\vec{n} = [3, 4, 5]$ passing through the point $P = (4, 5, 6)$.

b) (4 points) Parametrize the plane you found in a).

c) (3 points) Parametrize the line through P containing the vector $\vec{n}$.

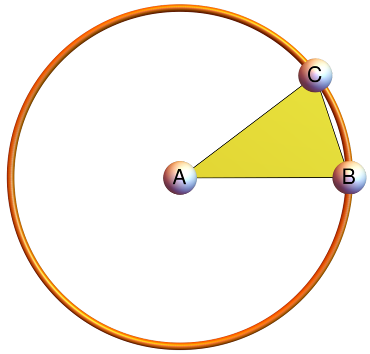
Solution:

a) $3x + 4y + 5z = 62$
b) For example: $r(s, t) = [4, 5, 6] + t[-4, -2, 4] + s[0, -5, 4]$ or $r(x, y) = [x, y, (62 - 3x - 4y)/5]$. c) $r(t) = [4, 5, 6] + t[3, 4, 5]$

Problem 5) (10 points)

a) (5 points) Use the cross product to verify that the triangle ABC with $A = (0, 0, 0)$, $B = (1, 0, 0)$ and $C = (\cos(\alpha), \sin(\alpha), 0)$ has area $\sin(\alpha)/2$.

b) (5 points) What is the distance of the point B to the line through A and C? The answer will depend of course on α.



Solution:

a) It is the length of the cross product $|\vec{AB} \times \vec{AC}|$.
b) Area/base = $\sin(t)$.

Problem 6) (10 points)

In the recent **top gun movie**, Maverick's fighter jet moves along the curve

$$\vec{r}(t) = [t, \sqrt{2}\frac{t^3}{3}, \frac{t^5}{5}] \ .$$

a) (5 points) Find the arc length which the jet traces from $t = -1$ to $t = 1$.

b) (5 points) What is the curvature of the curve at $t = 0$?

Solution:

a) 12/5
b) 0. To compute the curvature, use the formula $|r'(0) \times r''(0)|/|r'(0)|^3$. All the vectors $r'(0)[1, 0, 0]$ and $r''(0) = [0, 0, 0]$ are known.

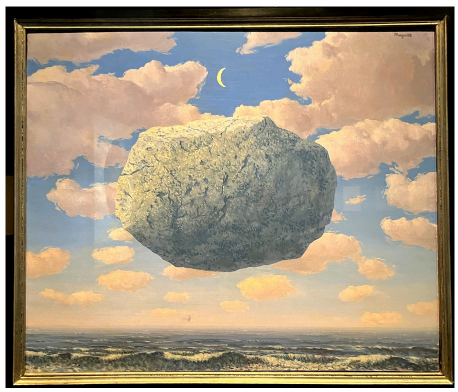
Problem 7) (10 points)

On a surreal world created by **René Margritte**, one can see a floating stone. This art could have motivated the **Haleluja mountains** in the Avatar movie. We of course all wait impatiently for the new Avatar 2022 movie.

Assume the stone has the property that it is subject to an acceleration

$$\vec{r}\,''(t) = [\cos(t), 0, 3 + \sin(1000t)]$$

and assume the stone is initially at position $\vec{r}(0) = [0, 0, 42]$ and that the initial velocity is $\vec{r}\,'(0) = [1, 0, 0]$. Find the path $\vec{r}(t)$ of the stone.



This photo of this Margritte picture was captured on June 16th by Oliver.

Solution:

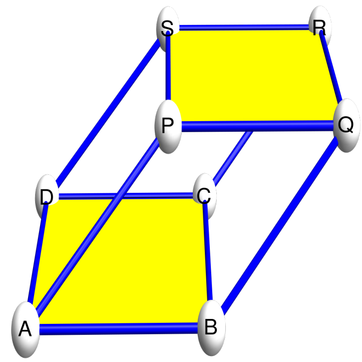
Fix the constants correctly. This was often done wrong. $[1 + t - \cos(t), 0, 42 + 3t^2/2 + t/1000 - \sin(1000t)/1000^2]$.

Problem 8) (10 points)

You know that the volume V of a parallel epiped with base points A, B, C, D and top points $P = A + \vec{u}, Q = B + \vec{u}, R = C + \vec{u}, S = D + \vec{u}$ is equal to $V = 10$. You also know that P has distance 5 to the base plane containing A, B, C, D. No other information is available to you.

a) (5 points) What is the area of the parallelogram $ABCD$?

b) (5 points) Compute the distance of the line through A, B and the line through Q, R.



Solution:

a) $A = 10/5 = 2$ b) $h = V/A = 10/2 = 5$

Problem 9) (10 points) No justifications are needed.

To make a nice summer picnic, we arrange an umbrella with chairs. We want to parametrize the surfaces which appear. Please do use the variables which are provided! It is part of the problem that you can work with parametrizations using all kind of variables:

a) (2 points) Parametrize the **floor plane** $z = 1$

$\vec{r}(s,t) = [\quad , \quad , \quad]$

b) (2 points) Parametrize a **decorative pole** $x^2 + y^2 = 1 + \sin(10z)/10$ holding the umbrella.

$\vec{r}(\theta,z) = [\quad , \quad , \quad]$

c) (2 points) Parametrize the healthy **Yoga chair** $x^2 + y^2 + 10z^2 = 1$.

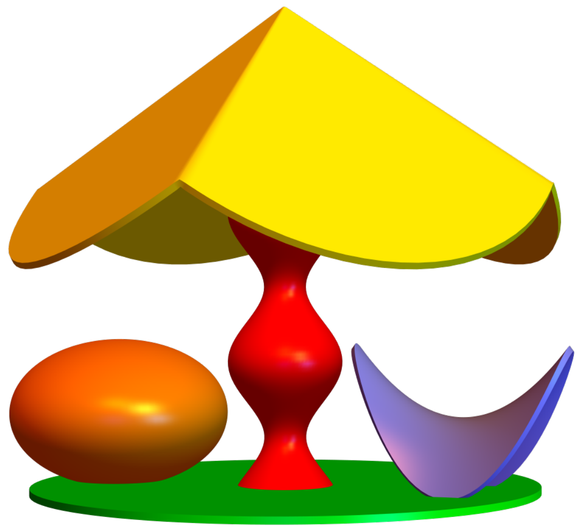
$\vec{r}(\theta,\phi) = [\quad , \quad , \quad]$

d) (2 points) The **umbrella hat** is given as the graph $z = 10 - |x| - |y|$.

$\vec{r}(x,y) = [\quad , \quad , \quad]$

e) (2 points) There is also a hyperbolic paraboloid $z = x^2 - y^2$ as an outdoor **"lazy chair"**.

$\vec{r}(u,v) = [\quad , \quad , \quad]$



Solution:

a) $r(s,t) = [s,t,0]$
b) $r(\theta,z) = [\sqrt{1+\sin(10z)/10}\cos(\theta), \sqrt{1+\sin(10z)/10}\sin(\theta), z]$.
c) $r(\theta,\phi) = [\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)/\sqrt{10}]$.
d) $r(x,y) = [x, y, 10 - |x| - |y|]$.
e) $r(u,v) = u, v, u^2 - v^2]$.