7/20/2023 SECOND HOURLY **Maths 21a, O.Knill, Summer 2023**

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) T F If $\nabla f(0,0)$ is non-zero then $\vec{v} = \nabla f(0,0)/|\nabla f(0,0)|$ is a unit vector for which $D_{\vec{v}}(f) = |\nabla f|^2$ is positive.

Solution:

Angles dance upwards.

2) T F The integral $\iint_R f(x,y)\, dA$ is the volume under the graph of a function and so always bigger or equal to zero.

Solution:

The function could be negative.

3) T F We can use multi-variable calculus to compute the improper integral $\int_{-\infty}^{\infty} e^{-x^2}\, dx$.

Solution:

Yes, this was the gifted problem

4) T F It is possible that a function $f(x,y)$ has a local maximum where $f_{xx} = -f_{yy}$.

Solution:

If $f_{xx} = -f_{yy}$ then D is negative so that we have a saddle point.

5) T F $(0,0)$ is a local maximum of the function $f(x,y) = 5 - x^{88} - y^{88}$.

Solution:

$(0,0)$ is a local maximum because the value there is 5 and the function is smaller everywhere else.

6) T **F**

Assume $(1,1)$ is a solution of the Lagrange equations for $f(x,y)$ under the constraint $g(x,y)=0$. If λ is negative then $(1,1)$ is a saddle point.

Solution:

The sign of λ has nothing to say about the nature of the critical point.

7) T **F**

The chain rule assures that $\frac{d}{dt}\vec{r}(f(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(f(t))$ if f is a function of three variables and $\vec{r}(t)$ parametrizes a curve.

Solution:

We can form $f(\vec{r}(t))$ but the combination $\vec{f}(\vec{r}(t))$ does not make sense.

8) **T** F

If R is the unit disk $x^2+y^2 \leq 1$, then $\iint_R x^2+y^2 \; dA \leq \iint_R 1 \; dA$.

Solution:

Yes, the integrand is always smaller or equal than 1.

9) **T** F

If $f(x,y,z) = x^2yz + y^3xz^2 + y^5 = 4$ defines z as a function $z = g(x,y)$ near the point $(1,1,1)$ then by implicit differentiation, $g_x(1,1) = -f_x(1,1,1)/f_z(1,1,1)$.

Solution:

By the formula $z_x = -f_x/f_z$.

10) T **F**

If $(1,1)$ is a critical point of f and $f_{xx}(1,1) = 1$ and $f_{yy}(1,1) = 1$, then $(1,1)$ is a local minimum.

Solution:

We do not know whether $D > 0$.

11) **T** F

If the discriminant at a critical point is non-zero, then we know the critical point is either a local maximum, a local minimum or a saddle point.

Solution:

This is the second derivative test

12) T **F** If $R = \{x > 0\}$ then $\iint_R x \, dxdy = \int_0^\pi \int_0^\infty r\cos(\theta) r dr d\theta$.

Solution:

The angle bound is false.

13) **T** F By linear approximation, we can estimate $\sqrt{102.4} = 10 + 2.4/20 = 10.12$.

Solution:

Yes, this is how we do linear estimation. The correct value is 10.1193.

14) **T** F If $(3,3)$ is a critical point of $f(x,y)$, then $(3,3)$ is also a critical point for the function $f(x,y)^2$.

Solution:

$\nabla f(x,y)^2 = 2f(x,y)\nabla f(x,y)$

15) T **F** The gradient of $f(x,y)$ is a vector perpendicular to the graph $z = f(x,y)$.

Solution:

The gradient of f is a vector with 2 components, not with 3 components.

16) **T** F If (x_0, y_0) is a min of $f(x,y)$ then (x_0, y_0) is a minimum under the constraint $g(x,y) = c$.

Solution:

Yes, $\nabla f(x,y) = 0$ automatically also gives a solution to the Lagrange equations.

17) T F The area of $x^4 + y^4 \leq 4$ is larger or equal than the area of the region $x^2 + y^2 \leq 4$.

Solution:

The first set contains the second set.

18) T F If $\vec{v}$ is a unit vector perpendicular to $\nabla f(1,1,1)$ and $(1,1,1)$ is not critical point of f, then $D_{\vec{v}} f(x,y,z) = 0$.

Solution:

We are perpendicular to the gradient.

19) T F The vector $\vec{r}_u(1,1)$ is tangent to the surface parametrized by $\vec{r}(u,v)$.

Solution:

These are velocity vectors of the grid curves.

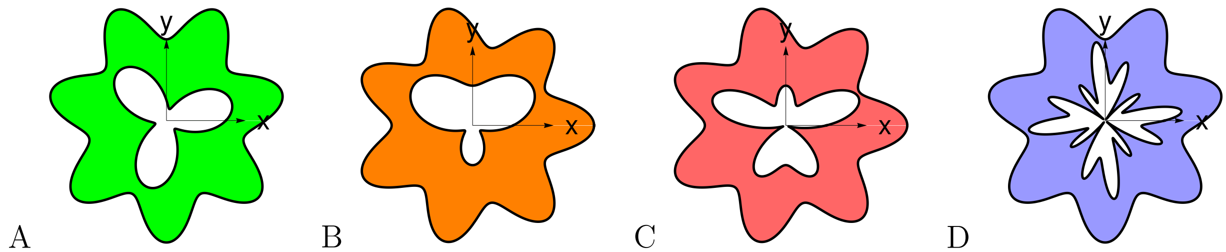
20) T F There is a function $f(x,y)$ which is continuous on the open square $(0,1) \times (0,1)$ but not continuous on $[0,1] \times [0,1]$ where Fubini fails.

Solution:

We have seen such an example in a slide during class.

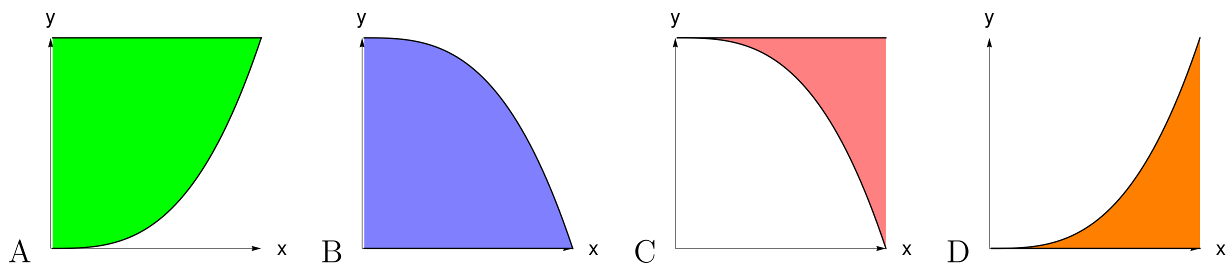
Problem 2) (10 points) No justifications are needed in this problem.

a) (4 points) Please match **polar regions** with area integrals. A-D are used exactly once.



Enter A-D	Area integral
	$\int_0^{2\pi} \int_{2+\sin(5t)+\sin(3t)}^{5+\cos(7t)} r\,dr d\theta$
	$\int_0^{2\pi} \int_{2+\sin(12t)+\cos(4t)}^{5+\sin(7t)} r\,dr d\theta$
	$\int_0^{2\pi} \int_{2+\sin(3t)+\sin(t)}^{5+\cos(7t)} r\,dr d\theta$
	$\int_0^{2\pi} \int_{2+\sin(3t)+\cos(3t)}^{5+\sin(7t)} r\,dr d\theta$

b) (4 points) Now match the regions with the area formulas. A-D are used exactly once.



Enter A-D	Area integral
	$\int_0^1 \int_{y^{1/3}}^1 1\,dxdy$
	$\int_0^1 \int_0^{1-x^3} 1\,dydx$
	$\int_0^1 \int_0^{y^{1/3}} 1\,dxdy$
	$\int_0^1 \int_{1-x^3}^1 1\,dydx$

c) (2 points) Recall the differential equations for the unknown function of two variables.

Wave equation for $f(t,x)$:

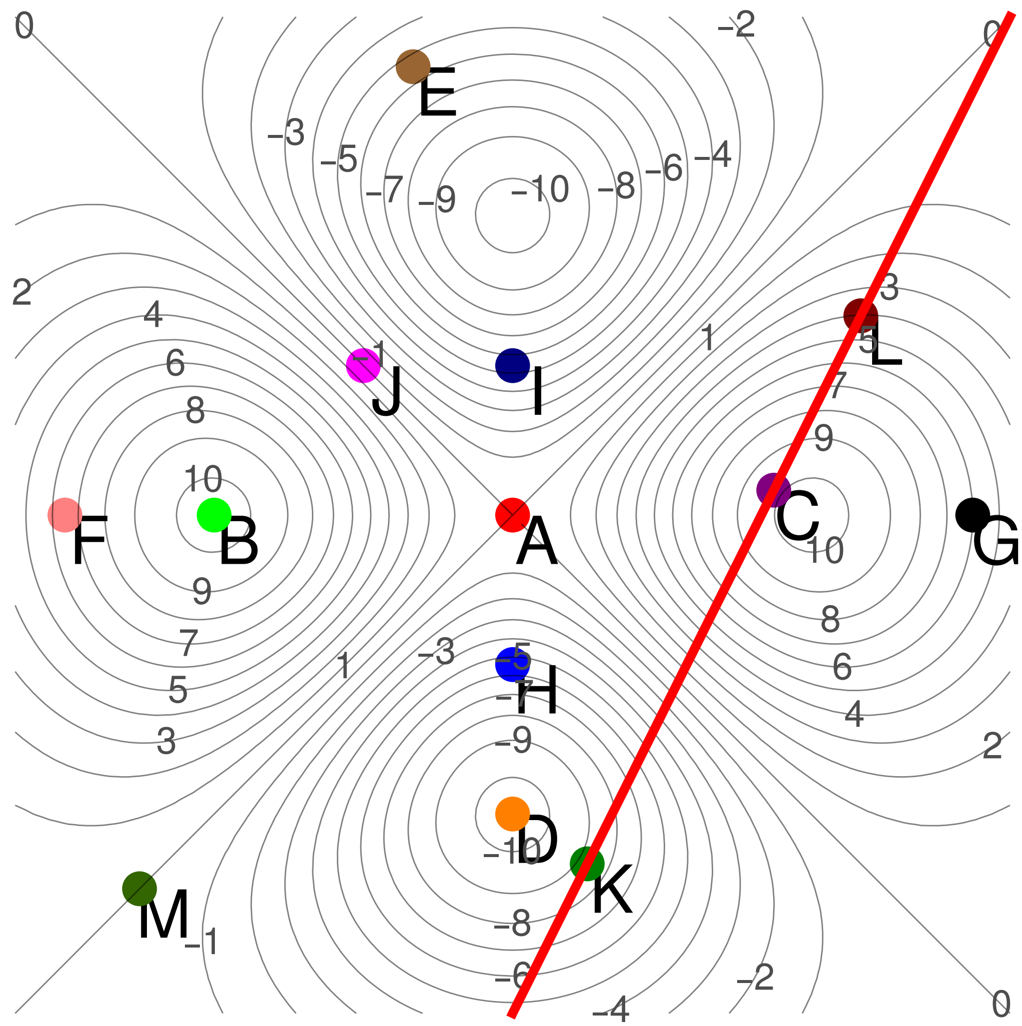
Laplace equation for $f(x,y)$:

Solution:

a) CDBA
b) DBAC
c) $f_{tt} = f_{xx}$ and $f_{xx} + f_{yy} = 0$.

Problem 3) (10 points) No justifications are needed in this problem.

Answer the following 10 questions for the **contour map** of a function $f(x, y)$. There is a line drawn which passes through the points K, C, L. This line is referred to as $g(x, y) = 0$ in the last two problems. Each problem is worth one point. Pick 10 of the 12 locations A-M on the map.



	Enter A-M
A point, where $f_x < 0, f_y = 0$.	
A point, where $f_x > 0, f_y = 0$.	
A point, where $f_y > 0, f_x = 0$.	
A point, where $f_y < 0, f_x = 0$.	
A saddle point of $f(x, y)$.	
A local minimum of $f(x, y)$.	
A local maximum of $f(x, y)$.	
A non critical point among A-M with minimal $\|\nabla f\| > 0$.	
A local maximum of $f(x, y)$ on $\{g(x, y) = 0\}$.	
A local minimum of $f(x, y)$ on $\{g(x, y) = 0\}$.	

Solution:

GFHI ADBMCK

Problem 4) (10 points)

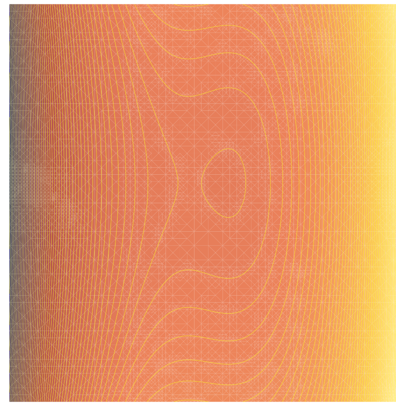
Elliptic curves are important in many fields like cryptography, analysis or physics.

a) (8 points) Classify the critical points of the function

$$f(x, y) = x^3 + y^2 - 3x - 2y$$

using the **second derivative test**.

b) (2 points) Is there a global max or min for f? If yes, locate them. Otherwise, tell why they do not exist.



Level curves of the function f are called elliptic curves.

Solution:

The gradient is $\nabla f(x, y) = [3x^2 - 3 = 0, 2y - 2 = 0]$. There are three critical points. The Hessian is

$$\begin{bmatrix} 6xy & 3x^2 - 1 \\ 3x^2 - 1 & 0 \end{bmatrix}$$

x	y	D	f_{xx}	Type	Value
-1	1	-12	-6	saddle	0
1	1	12	6	min	0

Problem 5) (10 points)

You have seen the analog of the problem for a dice where we had 6 variables. We do it here for a coin, where x is the probability that the coin shows head and y is the probability that the coin shows tail. Use the Lagrange method to solve this problem. Your task is to find the probability distribution (x, y) that maximizes the **Shannon entropy**

$$f(x, y) = -x\log(x) - y\log(y)$$

under the constraint $g(x, y) = x + y = 1$. As usual, we wrote $\log(x) = \ln(x)$ for the natural log.

Solution:

The Lagrange equations are

$$\begin{aligned} -1 - \log(x) &= \lambda \\ -1 - \log(y) &= \lambda \\ x + y &= 1 \end{aligned}$$

The solution is $(x, y) = (1/2, 1/2)$. It is the fair coin.

Problem 6) (10 points)

a) (3 points) Find the **tangent line** to the curve

$$f(x, y) = x^2 + xy + y^2 = 3$$

at the point $(1, 1)$.
b) (4 points) Find the **tangent plane** to the **air tag level surface**

$$f(x, y, z) = x^2 + xy + y^2 + yz + z^2 = 5 ,$$

at the point $(1, 1, 1)$.
c) (3 points) Estimate $f(1.01, 1.001, 1.0001)$ for the air tag function using linear approximation.

Oliver's wallet and keys are air tagged with ellipsoids

Solution:

a) $x + y = 2$.
b) $3x + 4y + 3z = 10$
c) 5.0343

Problem 7) (10 points)

Use the standard surface area formula to compute the surface area of the surface which has the parametrization

$$\vec{r}(u,v) = \begin{bmatrix} u^2 + v^2 \\ u^2 - v^2 \\ u^2 \end{bmatrix}$$

and where the parameter domain is $0 \leq u \leq 1$ and $0 \leq v \leq 1$.

Solution:

The $|r_u \times r_v|^2 = 96u^2v^2$ gives $|r_u \times r_v| = uv\sqrt{6}/4$ The result after integration is $\sqrt{6}$.

Problem 8) (10 points)

In order to honor the magic of the number 21, we evaluate the double integral

$$\int_1^2 \int_0^{4-x^2} \frac{y^{21}}{\sqrt{4-y}-1} \, dy \, dx \; .$$

Solution:

Standard change of order of integration problem. The answer is $3^{22}/22$.

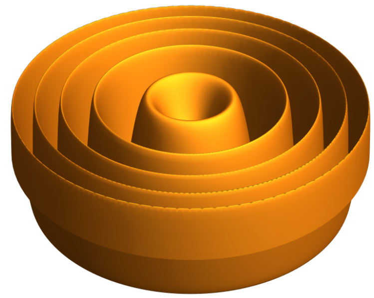
Problem 9) (10 points)

Evaluate the double integral

$$\iint_G 2\pi \sin(\pi(x^2+y^2))\, dxdy \ ,$$

where G is region given by

$$\{0 \leq x^2+y^2 \leq 9, y > |x|\} \ .$$



The graph of $2\pi \sin(\pi(x^2+y^2))$.

Solution:

Use polar coordinates. The final result is π.