

# MULTIVARIABLE CALCULUS

MATH S-21A

## Unit 2: Vectors and dot product

### LECTURE

**2.1.** Two points  $P = (a, b, c)$  and  $Q = (x, y, z)$  in space  $\mathbf{R}^3$  define a **vector**  $\vec{v} = \begin{bmatrix} x - a \\ y - b \\ z - c \end{bmatrix}$ . We write this column vector also as a row vector  $[x - a, y - b, z - c]$  for typographic reasons. As the vector starts at  $P$  to  $Q$  we write  $\vec{v} = \vec{PQ}$ . The real numbers  $p, q, r$  in  $\vec{v} = [p, q, r]$  are called the **components** of  $\vec{v}$ .

**2.2.** Vectors can be attached to any point in space. Two vectors with the same components are considered **equal** as they can be translated into each other. If a vector  $\vec{v}$  starts at the origin  $O = (0, 0, 0)$ , then  $\vec{v} = [p, q, r]$  heads from  $O$  to the point  $P = (p, q, r)$ . One can therefore identify **points**  $P = (a, b, c)$  with **vectors**  $\vec{v} = [a, b, c]$  attached to the origin. For clarity reasons, we often draw an arrow  $\vec{\phantom{v}}$  on top of a vector variable and if  $\vec{v} = \vec{PQ}$  then  $P$  is the “tail” and  $Q$  is the “head” of the vector. To distinguish vectors from points, it is custom to write  $[2, 3, 4]$  for vectors and  $(2, 3, 4)$  for points.<sup>1</sup>

**2.3.**

**Definition:** The **sum** of two vectors is  $\vec{u} + \vec{v} = [u_1, u_2] + [v_1, v_2] = [u_1 + v_1, u_2 + v_2]$ . The **scalar multiple** is  $\lambda\vec{u} = \lambda[u_1, u_2] = [\lambda u_1, \lambda u_2]$ . The **difference**  $\vec{u} - \vec{v}$  can best be seen as the addition of  $\vec{u}$  and  $(-1) \cdot \vec{v}$ .

**Commutativity, associativity, or distributivity** rules for vectors are inherited from the corresponding rules for numbers. Please review these rules yourself if necessary.

**2.4.** The vectors  $\vec{i} = [1, 0, 0]$ ,  $\vec{j} = [0, 1, 0]$ ,  $\vec{k} = [0, 0, 1]$  are called **standard basis vectors**. This has historically grown because the dot and cross product have grown from **quaternions** which are points  $(t, x, y, z)$  in  $\mathbb{R}^4$ , usually written as  $q = t + ix + jy + kz$ .

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<sup>1</sup>We avoid the Stewart notation  $\langle p, q, r \rangle$ , which is harder to read. Programming languages like Python use  $[p, q, r]$ . Mathematica, Perl or C use  $\{p, q, r\}$ . In linear algebra it is useful to distinguish row and column vectors.

2.5.

**Definition:** The **length**  $|\vec{v}|$  of a vector  $\vec{v} = \vec{PQ}$  is defined as the distance  $d(P, Q)$  from  $P$  to  $Q$ . A vector of length 1 is called a **unit vector**. If  $\vec{v} \neq \vec{0}$ , then  $\vec{v}/|\vec{v}|$  is called a **direction** of  $\vec{v}$ . The only vector of length 0 is the zero vector  $\vec{0} = [0, 0, 0]$ .

2.6.

**Definition:** The **dot product** of two vectors  $\vec{v} = [a, b, c]$  and  $\vec{w} = [p, q, r]$  is defined as  $\vec{v} \cdot \vec{w} = ap + bq + cr$ .

2.7. Different notations for the dot product are used in different mathematical fields. While mathematicians write  $\vec{v} \cdot \vec{w}$  or  $(\vec{v}, \vec{w})$  or  $\langle \vec{v}, \vec{w} \rangle$ , the **Dirac notation**  $\langle \vec{v} | \vec{w} \rangle$  is used in quantum mechanics. The **Einstein notation**  $v_i w^i$  or more generally  $g_{ij} v^i w^j$  is used in general relativity. In statistics, it is called the **covariance**  $\text{Cov}[v, w]$  of centered data. The dot product is also called **scalar product** or **inner product**. It could be generalized. Any product  $g(v, w)$  which is bi-linear (linear both in  $v$  and  $w$ ) and satisfies the symmetry  $g(v, w) = g(w, v)$  and  $g(v, v) \geq 0$  and  $g(v, v) = 0$  if and only if  $v = 0$ , can be used as a dot product. An example is  $g(v, w) = 2v_1 w_1 + 3v_2 w_2 + 5v_3 w_3$ .

2.8. The dot product determines distances. A bit surprising, distances determine the dot product: **Proof:** Express the length of  $\vec{v}$  as  $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$ . On the other hand, from  $(\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2(\vec{v} \cdot \vec{w})$  can be solved for  $\vec{v} \cdot \vec{w}$ :

$$\vec{v} \cdot \vec{w} = (|\vec{v} + \vec{w}|^2 - |\vec{v}|^2 - |\vec{w}|^2)/2 .$$

2.9. The **Cauchy-Schwarz inequality** is

**Theorem:**  $|\vec{v} \cdot \vec{w}| \leq |\vec{v}| |\vec{w}|$ .

*Proof.* If  $|\vec{w}| = 0$ , the statement holds because both sides are zero. Otherwise, assume  $|\vec{w}| = 1$  by dividing the equation by  $|\vec{w}|$ . Now plug in  $a = \vec{v} \cdot \vec{w}$  into the equation  $0 \leq (\vec{v} - a\vec{w}) \cdot (\vec{v} - a\vec{w})$  to get  $0 \leq (\vec{v} - (\vec{v} \cdot \vec{w})\vec{w}) \cdot (\vec{v} - (\vec{v} \cdot \vec{w})\vec{w}) = |\vec{v}|^2 + (\vec{v} \cdot \vec{w})^2 - 2(\vec{v} \cdot \vec{w})^2 = |\vec{v}|^2 - (\vec{v} \cdot \vec{w})^2$  which means  $(\vec{v} \cdot \vec{w})^2 \leq |\vec{v}|^2$ .  $\square$

2.10. Having established this, it is possible to give, without referring to geometric pictures, a definition of what an **angle** is:

**Definition:** The **angle** between two nonzero vectors  $\vec{v}, \vec{w}$  is defined as the unique  $\alpha \in [0, \pi]$  which satisfies  $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$ . Since  $\cos$  maps  $[0, \pi]$  in a 1:1 manner to  $[-1, 1]$ , this is well defined.

2.11. The **Al Kashi theorem** gives the third side length  $c$  of a triangle  $ABC$  in terms of the sides  $a = d(A, C), b = d(B, C)$  and  $\alpha$ , the angle located at the vertex  $A$ .

**Theorem:**  $a^2 + b^2 = c^2 - 2ab \cos(\alpha)$ .

*Proof.* Define  $\vec{v} = \vec{AB}$ ,  $\vec{w} = \vec{AC}$ . Because  $c^2 = |\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{v} \cdot \vec{w}$ , We know  $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$  so that  $c^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}| \cdot |\vec{w}| \cos(\alpha) = a^2 + b^2 - 2ab \cos(\alpha)$ .  $\square$

**2.12.** The **triangle inequality** tells

**Theorem:**  $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$

*Proof.*  $|\vec{u} + \vec{v}|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u}^2 + \vec{v}^2 + 2\vec{u} \cdot \vec{v} \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u} \cdot \vec{v}| \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u}| \cdot |\vec{v}| = (|\vec{u}| + |\vec{v}|)^2$ .  $\square$

**Definition:** Two vectors are called **orthogonal** or **perpendicular** if  $\vec{v} \cdot \vec{w} = 0$ . The zero vector  $\vec{0}$  is orthogonal to any vector. For example,  $\vec{v} = [2, 3]$  is orthogonal to  $\vec{w} = [-3, 2]$ .

**2.13.** We can now prove the **Pythagorean theorem**:

**Theorem:** If  $\vec{v}$  and  $\vec{w}$  are orthogonal, then  $|\vec{v} - \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$ .

*Proof.*  $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w}$ . We usually write this as  $a^2 + b^2 = c^2$ .  $\square$

**2.14.**

**Definition:** The vector  $P(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \vec{w}$  is called the **projection** of  $\vec{v}$  onto  $\vec{w}$ . The **scalar projection**  $\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|}$  is a **signed length** of the vector projection. Its absolute value is the length of the projection of  $\vec{v}$  onto  $\vec{w}$ . The vector  $\vec{b} = \vec{v} - P(\vec{v})$  is a vector orthogonal to the  $\vec{w}$ -direction.

**2.15.** The projection allows to **visualize** the dot product. The absolute value of the dot product is the length of the projection. The dot product is positive if  $\vec{v}$  points more towards to  $\vec{w}$ , it is negative if  $\vec{v}$  points away from it. In the next class, we use the projection to compute distances between various objects.

#### EXAMPLES

**2.16.** For example, with  $\vec{v} = [0, -1, 1]$ ,  $\vec{w} = [1, -1, 0]$ ,  $P(\vec{v}) = [1/2, -1/2, 0]$ . Its length is  $1/\sqrt{2}$ .

**2.17.** The **RGB color space** consists of triples  $\vec{v} = [r, g, b]$  describing the amount of red, green and blue of a **color**. An other coordinate system is the **CMY color space** consisting of triples  $\vec{v} = [c, m, y] = [1 - r, 1 - g, 1 - b]$ , where  $c$  is **cyan**,  $m$  is **magenta** and  $y$  is **yellow**.

**2.18.** In physics, forces and fields  $\vec{F}$  are described by vectors. The **velocity** of a curve  $r(t) = [x(t), y(t), z(t)]$  is a vector attached to the point  $r(t)$ . We will come to this.

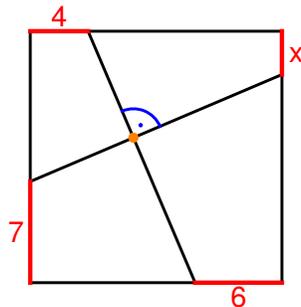
**2.19.** In probability theory, data are described by vectors. One calls them also **random variables**. It is in statistics, where higher dimensional spaces appear. In statistics,  $\vec{v} \cdot \vec{w}$  is the **covariance** and  $\cos(\alpha) = (\vec{v} \cdot \vec{w}) / (|\vec{v}||\vec{w}|)$  is the **correlation**.

HOMEWORK

This homework is due on Tuesday, 7/02/2024.

**Problem 2.1:** a) Find a **unit vector** parallel to  $\vec{x} = \vec{u} + \vec{v} + \vec{w}$  if  $\vec{u} = [4, 3, 1]$  and  $\vec{v} = [2, 4, 1]$  and  $\vec{w} = [0, 0, 4]$ .  
 b) Verify that in a right angle triangle, the relation  $1/a^2 + 1/b^2 = 1/h^2$  holds, where  $h$  is the height perpendicular to the hypotenuse  $c$ .

**Problem 2.2:** Find  $x$  in the following picture about a square. The riddle appeared in a German magazine. Please document how you think about it and also record also attempts which failed.



**Problem 2.3: Colors** are encoded by vectors  $\vec{v} = [\text{red}, \text{green}, \text{blue}]$ . The red, green and blue components of  $\vec{v}$  are all real numbers in the interval  $[0, 1]$ .  
 a) Determine the angle between the colors yellow and magenta.  
 b) What is the vector projection of the magenta-orange mixture  $\vec{x} = (\vec{v} + \vec{w})/2$  onto green  $\vec{y}$ ?

**Problem 2.4:** A rope is wound exactly 8 times around a stick of circumference 1 and length 15. How long is the rope?

**Problem 2.5:** a) Find the angle between the main diagonal of the unit cube and one of the face diagonals. Assume that both diagonals pass through a common vertex.  
 b) Find the vector projection of the main diagonal  $\vec{v} = [1, 1, 1]$  onto the side diagonal  $\vec{w} = [1, 1, 0]$ .  
 c) Find the maximal distance between the 16 points  $(\pm 1, \pm 1, \pm 1, \pm 1)$  of a **tesseract**.