MULTIVARIABLE CALCULUS

MATH S-21A

Unit 9: Partial derivatives

Lecture

9.1. If a functions depends on several variables we can differentiate it with respect to any of the variables:

Definition: If $f(x, y)$ is a function of the two variables x and y, then the **partial derivative** $\frac{\partial}{\partial x} f(x, y)$ is defined as the derivative of the function $g(x) = f(x, y)$ with respect to x, where y is kept to be a constant. The partial derivative with respect to y is the derivative with respect to y, where x is fixed.

9.2. The short hand notation $f_x(x, y) = \frac{\partial}{\partial x} f(x, y)$ is convenient. When iterating derivatives, the notation is similar: we write for example $f_{xy} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f$. The number $f_x(x_0, y_0)$ gives the slope of the graph sliced at (x_0, y_0) in the x-direction. The second derivative $f_{xx}(x_0, y_0)$ is a measure of concavity in the x-direction at (x_0, y_0). The meaning of $f_{xy}(x_0, y_0)$ is the rate of change of the x-slope.

9.3. The notation $\partial_x f, \partial_y f$ was introduced by Carl Gustav Jacobi. Before that, Josef Lagrange used the term "partial differences". For functions of three or more variables, the partial derivatives are defined in the same way. We write for example $f_x(x, y, z)$ or $f_{xxz}(x, y, z)$.

Theorem: Clairaut's theorem: If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

9.4. Proof. Following Euler, we first look at the **difference quotients**. If the "Planck constant" h is positive, define $f_x(x, y) = [f(x + h, y) - f(x, y)]/h$. The limit $h = 0$, is then the definition of the partial derivative f_x. Comparing the two sides of the equation for fixed $h > 0$ shows:

$$hf_x(x, y) = f(x + h, y) - f(x, y) \qquad hf_y(x, y) = f(x, y + h) - f(x, y).$$

$$h^2 f_{xy}(x, y) = f(x+h, y+h) - f(x, y+h) - (f(x+h, y) - f(x, y)) \qquad h^2 f_{yx}(x, y) = f(x+h, y+h) - f(x+h, y) - (f(x, y+h) - f(x, y))$$

9.5. Without having taken any limits, we established an identity which holds for all $h > 0$: the discrete derivatives f_x, f_y satisfy the relation $f_{xy} = f_{yx}$ for any $h > 0$. We could fancy it as **"quantum Clairaut" formula**. If the classical derivatives f_{xy}, f_{yx} are both continuous, it is possible to take the limit $h \to 0$. The classical Clairaut's theorem can then be seen as a "classical limit". The quantum Clairaut holds however for **all** functions $f(x, y)$ of two variables. Not even continuity is needed. [1]

9.6. An equation for an unknown function $f(x, y)$ which involves partial derivatives with respect to at least two different variables is called a **partial differential equation**. We abbreviate PDE. If only the derivative with respect to one variable appears, it is an **ordinary differential equation**. This is abbreviated as an ODE.

Examples

9.7. For $f(x, y) = x^4 - 6x^2y^2 + y^4$, we have $f_x(x, y) = 4x^3 - 12xy^2$, $f_{xx} = 12x^2 - 12y^2$, $f_y(x, y) = -12x^2y + 4y^3$, $f_{yy} = -12x^2 + 12y^2$ and see that $\Delta f = f_{xx} + f_{yy} = 0$. A function which satisfies $\Delta f = 0$ is also called **harmonic**. The equation $f_{xx} + f_{yy} = 0$ is a PDE:

Definition: A **partial differential equation** (PDE) is an equation for an unknown function $f(x, y)$ which involves partial derivatives with respect to more than one variables.

9.8.

The **wave equation** $f_{tt}(t, x) = f_{xx}(t, x)$ governs the motion of light or sound. The function $\boxed{f(t, x) = \sin(x - t) + \sin(x + t)}$ satisfies the wave equation.

The **heat equation** $f_t(t, x) = f_{xx}(t, x)$ can be used to model diffusion of heat or the spread of an epidemic. The function $\boxed{f(t, x) = \frac{1}{\sqrt{t}} e^{-x^2/(4t)}}$ satisfies the heat equation.

The **Laplace equation** $f_{xx} + f_{yy} = 0$ determines the shape of a membrane. The function $\boxed{f(x, y) = x^3 - 3xy^2}$ is an example satisfying the Laplace equation.

The **advection equation** $f_t = f_x$ is used to model transport in a wire. The function $\boxed{f(t, x) = e^{-(x+t)^2}}$ satisfies the advection equation.

[1] For a more detailed proof of Clairaut's theorem, see *www.math.harvard.edu/knill/teaching/math22a2018/handouts/lecture14.pdf*.

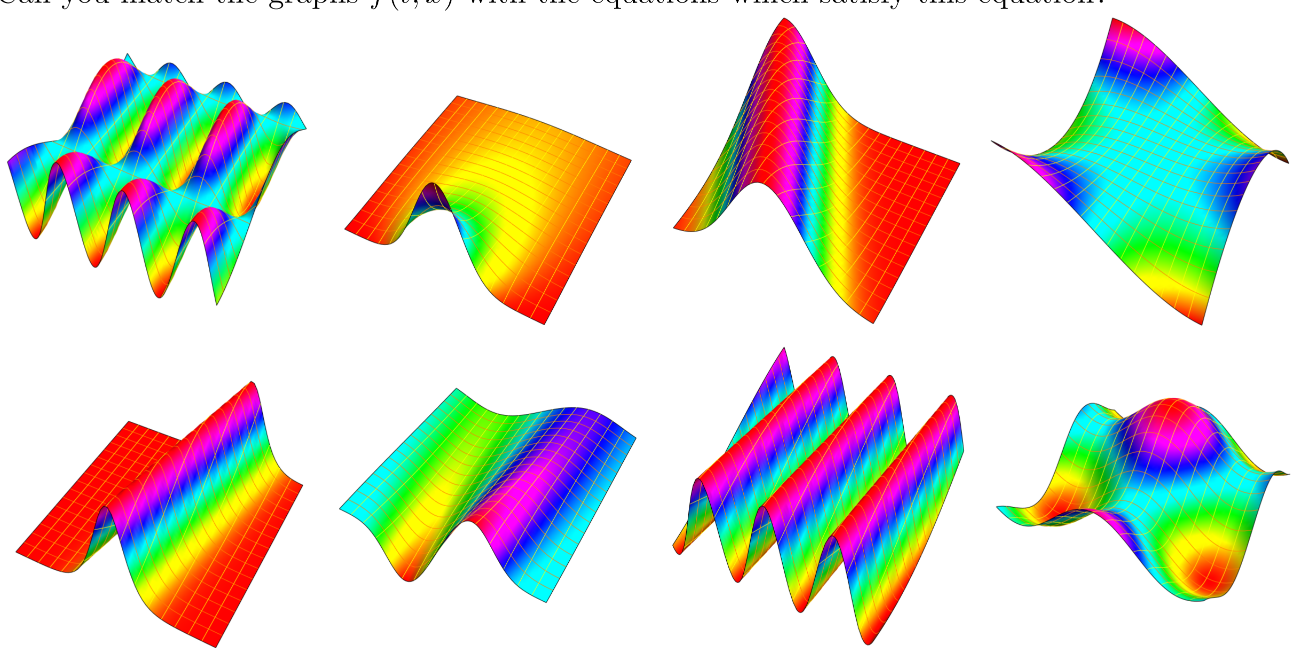
The **Burgers equation** $f_t + f f_x = f_{xx}$ describes waves at the beach which break. The function $f(t,x) = \frac{x}{t} \frac{\sqrt{\frac{1}{t}} e^{-x^2/(4t)}}{1+\sqrt{\frac{1}{t}} e^{-x^2/(4t)}}$ satisfies the Burgers equation.

The **eiconal equation** $f_x^2 + f_y^2 = 1$ is used to see the evolution of wave fronts in optics. The function $f(x,y) = \cos(x) + \sin(y)$ satisfies the eiconal equation.

The **KdV equation** $f_t + 6 f f_x + f_{xxx} = 0$ models **water waves** in a narrow channel. The function $f(t,x) = \frac{a^2}{2} \cosh^{-2}(\frac{a}{2}(x - a^2 t))$ satisfies the KdV equation.

The **Schrödinger equation** $f_t = \frac{i\hbar}{2m} f_{xx}$ is used to describe a **quantum particle** of mass m. The function $f(t,x) = e^{i(kx - \frac{\hbar}{2m}k^2 t)}$ solves the Schrödinger equation. [Here $i^2 = -1$ is the imaginary i and $\hbar$ is the **Planck constant** $\hbar \sim 10^{-34} Js$.]

Can you match the graphs $f(t,x)$ with the equations which satisfy this equation?



9.9. In all examples, we just see one possible solution to the partial differential equation. There are in general many solutions and additional initial or boundary conditions then determine the solution uniquely. If we know $f(0,x)$ for the Burgers equation, then the solution $f(t,x)$ is determined.

Homework

This homework is due on Tuesday, 7/16/2024 in class.

Problem 9.1: Verify that $f(t,x) = 2\exp((t+x)^2) + 4\sin(\cos(t+x))$ is a solution of the transport equation $f_t(t,x) = f_x(t,x)$.

Problem 9.2: a) Verify that $f(x,y) = \sin(2x)(\cos(3y)+\sin(3y))$ satisfies the **Klein Gordon equation** $u_{xx} - u_{yy} = 5u$. Try without technology first. This PDE is useful in quantum mechanics.
b) Verify that $4\arctan(e^{(x-\frac{t}{2})\frac{2}{\sqrt{3}}})$ satisfies the **Sin-Gordon equation** $u_{tt} - u_{xx} = -\sin(u)$. Use might here want to use technology.

Problem 9.3: Verify that for any real constant b, the function $f(x,t) = e^{-bt}\cos(x+t)$ satisfies the **driven transport equation** $f_t(x,t) = f_x(x,t) - bf(x,t)$ This PDE is sometimes called the **advection equation** with damping factor b.

Problem 9.4: The differential equation

$$f_t = f - xf_x - x^2 f_{xx}$$

is a version of the **infamous Black-Scholes equation**. Here $f(x,t)$ is the prize of a **call option** and x the stock prize and t is time. Find a function $f(x,t)$ solving it which depends both on x and t. The solutions $f(x,t) = x$ or $f(x,t) = e^t$ which only depends on one variable can help you to get inspired.

Problem 9.5: The partial differential equation $f_t + ff_x = f_{xx}$ is called **Burgers equation** and describes waves at the beach. In higher dimensions, it leads to the **Navier-Stokes equation** which are used to describe the weather. Verify that

$$f(t,x) = \frac{\left(\frac{1}{t}\right)^{3/2} x e^{-\frac{x^2}{4t}}}{\sqrt{\frac{1}{t}} e^{-\frac{x^2}{4t}} + 1}$$

solves the Burgers equation. You also here might want to get help with technology.

Oliver Knill, knill@math.harvard.edu, Math S-21a, Harvard Summer School, 2024