7/25/2024 SECOND HOURLY Practice 2 **Maths 21a, O.Knill, Summer 2024**

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) T F The integral $\iint_R 1 \, dA$ is the area of a two dimensional region R.

Solution:

A very important fact. This is how area should be written.

2) T F If $x^4 y + y^3 x = 2$ defines y as a function of x then by implicit differentiation, $y'(1) = -5/4$.

Solution:

By the formula $y' = -f_x/f_y$.

3) T F If $f_{xx} > 0$ and the discriminant $D > 0$, then $f_{yy} < 0$.

Solution:

If they would have different signs, the discriminant would be zero.

4) T F $(0,0)$ is neither a max nor min of the function $f(x,y) = x^4 + y^4$ because the discriminant D is zero there.

Solution:

We can have max or min despite D=0

5) T F If $R = \{x^2 + y^2 \leq 9\}$ then $\iint_R x^2 + y^2 \, dxdy = \int_0^{2\pi} \int_0^3 r^2 \, dr d\theta$.

Solution:

The integration factor r was missing.

6) T F The value of the function $f(x,y) = \sin(2x+4y)$ at $(x,y) = (0.01, -0.01)$ can by linear approximation near $(x_0, y_0) = (0,0)$ be estimated as -0.02.

Solution:

Indeed by linear approximation of $\sin(2x+4y)$ is $L(x,y) = 2x + 4y$.

7) T F If $(1,1)$ is a critical point of $f(x,y)$, then $(1,1)$ is also a critical point for the function $g(x,y) = f(x^3, y)$.

Solution:

If $\nabla f(1,1) = (f_x(1,1), f_y(1,1)) = (0,0)$ then also $\nabla g(1,1) = (f_x(1,1)3x^2, f_y(1,1)) = (0,0)$.

8) T F The gradient of $f(x,y)$ is normal to the level curves of f.

Solution:

It is in general true.

9) T F If (x_0, y_0) is a max of $f(x,y)$ under the constraint $g(x,y) = g(x_0, y_0)$, then (x_0, y_0) is a max of $g(x,y)$ under some constraint $f(x,y) = f(x_0, y_0)$.

Solution:

it could be a min for example.

10) T F The area of a filled ellipse $x^2/4 + y^2/9 \leq 4$ is equal to 4 times the area of the filled ellipse $x^2/4 + y^2/9 \leq 1$.

Solution:

Several possibilities to solve this: 1. Compute the areas either by integration or knowing πab for an ellipse. 2. look at the geometry: the first ellipse is twice as long, and twice as high. This change the area by 4.

11) T F

If $\vec{v}$ is a unit vector parallel to the surface $f(x, y, z) = 0$ $D_{\vec{v}} f(x, y, z) = 0$.

Solution:

By definition $D_{\vec{v}} f(x, y, z) = \nabla f \cdot \vec{v}$. Because ∇f is perpendicular to the surface and $\vec{v}$ by assumption is parallel, we have the statement.

12) T F

If $\vec{r}(t) = [x(t), y(t)]$ and $x(t), y(t)$ are non-constant polynomials like $x(t) = 1 + t^2, y(t) = 1 + t^3$ then the unit tangent vector is defined at all points.

Solution:

Take the example $\vec{r}(t) = [t^2, t^3]$. At $t = 0$, we have a cusp.

13) T F

The vector $\vec{r}_v(u, v)$ is tangent to the surface parameterized by $\vec{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$.

Solution:

The vector $\vec{r}_u$ is tangent to a grid curve and so tangent to the surface.

14) T F

The second derivative test involving D and f_{xx} can check whether a Lagrange multiplier solution of f under the constraint $g = c$ is a max or min.

Solution:

No, the second derivative test applies for function $f(x, y)$ without constraint.

15) T F

If $(0, 0)$ is a critical point of $f(x, y)$ and $D = 0$ but $f_{xx}(0, 0) > 0$ then $(0, 0)$ is not a local max.

Solution:

If $f_{xx}(0, 0) > 0$ then on the x-axis the function $g(x) = f(x, 0)$ has a local minimum. This means that there are points close to $(0, 0)$ where the value of f is larger.

16) T F

Let (x_0, y_0) be a saddle point of $f(x, y)$. For any unit vector $\vec{u}$, there are points arbitrarily close to (x_0, y_0) for which ∇f is parallel to $\vec{u}$.

Solution:

Just look at the level curves near a saddle point. The gradient vectors are orthogonal to the level curves which are hyperbola. You see that they point in any direction except 4 directions. To see this better, take a pen and draw a circle around the saddle point between two of your knuckles on your fist. At each point of the circle, now draw the direction of steepest increase (this is the gradient direction).

17) T F

If $f(x, y)$ has two max, then it must have a min.

Solution:

Look at a camel type surface $xe^{-x^2-y^2}$. There is no local minimum.

18) T F

Given a unit vector $\vec{v}$ and (x_0, y_0) a critical point of f, then $D_{\vec{v}} f(x_0, y_0) > 0$ at a local minimum.

Solution:

The directional derivative is zero because the gradient is zero.

19) T F

The chain rule assures that $d/dt(x(t)^2 + y(t)^2)) = 2x(t)x'(t) + 2y(t)y'(t)$.

Solution:

Yes this is correct. Just written out what the chain rule means.

20) T F

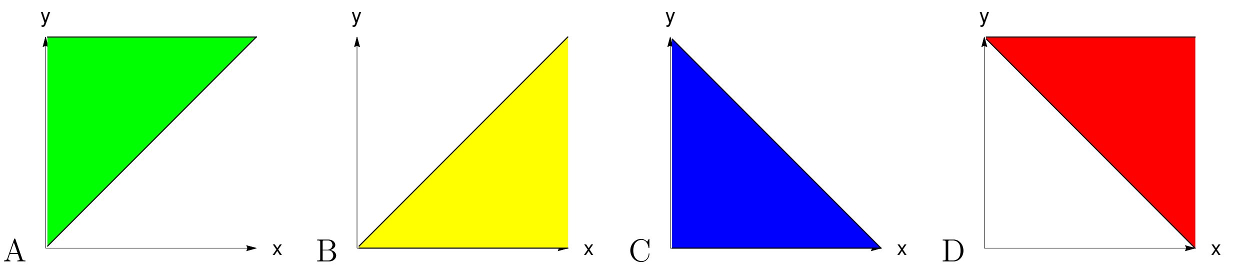
The critical points of $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$ are solutions to the Lagrange equations when extremizing the function $f(x, y)$ under the constraint $g(x, y) = 0$.

Solution:

The critical points of F are points where $f_x = \lambda g_x, f_y = \lambda g_y, g = 0$ which is exactly the Lagrange equations.

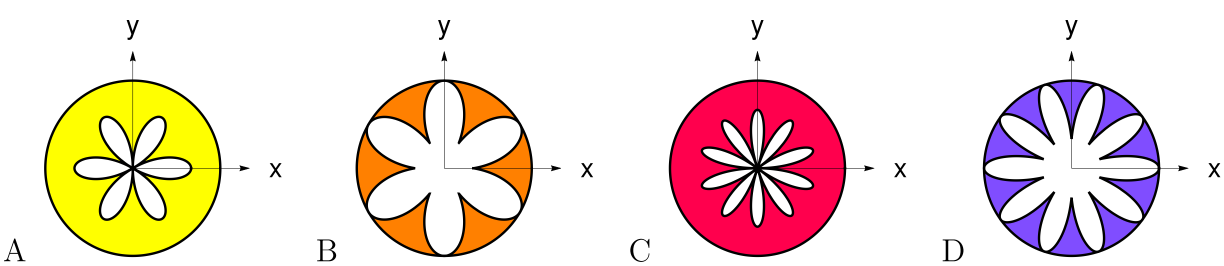
Problem 2) (10 points) No justifications are needed in this problem.

a) (4 points) Match the regions with the area formulas. A-D are used exactly once.



Enter A-D	Area integral
	$\int_0^1 \int_y^1 1 \; dxdy$
	$\int_0^1 \int_0^{1-x} 1 \; dydx$
	$\int_0^1 \int_0^y 1 \; dxdy$
	$\int_0^1 \int_{1-x}^1 1 \; dydx$

b) (4 points) Now match polar regions with area integrals. A-D are used exactly once.



Enter A-D	Area integral
	$\int_0^{2\pi} \int_{1+2\|\sin(3\theta)\|}^3 r dr d\theta$
	$\int_0^{2\pi} \int_{1+2\|\cos(5\theta)\|}^3 r dr d\theta$
	$\int_0^{2\pi} \int_{2\|\sin(5\theta)\|}^3 r dr d\theta$
	$\int_0^{2\pi} \int_{2\|\cos(3\theta)\|}^3 r dr d\theta$

c) (2 points) Write down the differential equations for the unknown function $f(t, x)$.

Transport equation:

Heat equation:

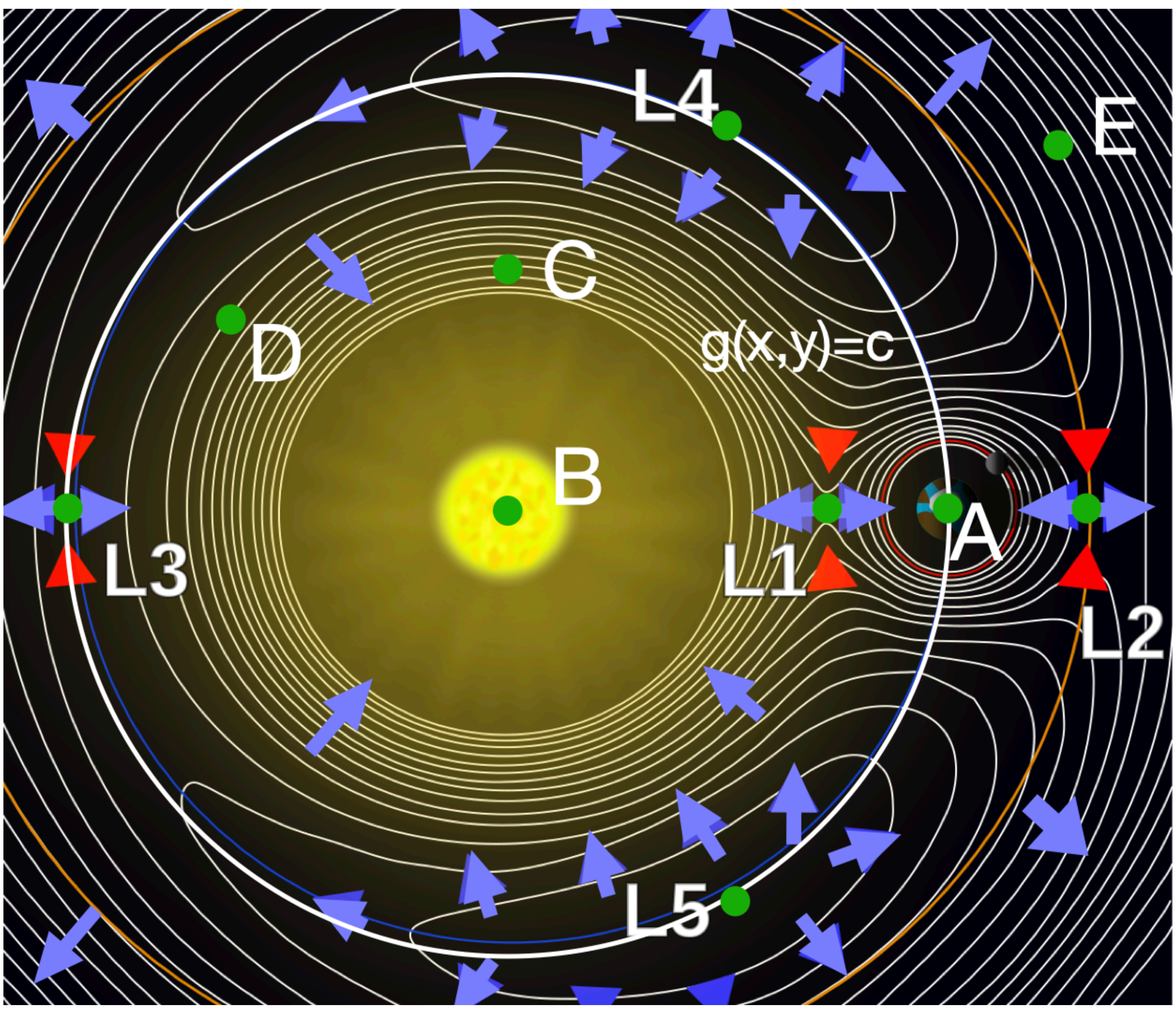
Solution:

a) BCAD
b) BDCA
c) $f_t = f_x$
d) $f_t = f_{xx}$

Problem 3) (10 points) No justifications are needed in this problem.

(10 points) We all have seen the nice pictures of the **Webb telescope**. It sits at the **Lagrange point** L_2 on the line through the sun (B) and earth (A). We modified a Wikipedia picture explaining the Lagrange points and added a few more points $A-E$. The contour map visualizes the effective potential of the rotating system in which the sun-earth axis is at rest. The vectors you see represent gradients $\nabla f(x,y)$. The circle through A, L_3, L_4, L_5 centered at the sun B is written here as $g(x,y)=c$. Each of the questions is worth 1 point.

	Enter A-E,L_1,...,L_5 here
Which points are minima of f?	
Which points are maxima of f?	
Which points are saddles?	
Point with maximal $\|\nabla f\|$.	
Points with $f_x > 0$.	
Points with $f_x < 0$.	
Points with $f_y > 0$.	
Points with $f_y < 0$.	
Points, where f is maximal on $g(x,y)=c$.	
Points, where f is minimal on $g(x,y)=c$.	



Solution:

L4,L5
A,B
L1,L2,L3
C
D,E
-
E
C,D
A,L3
L4,L5

Problem 4) (10 points)

Classify the critical points of the function

$$f(x, y) = x^3y - xy$$

using the **second derivative test**. Is there a global max or min?

Solution:

The gradient is $\nabla f(x, y) = [3x^2y - y, x^3 - x]$. There are three critical points. The Hessian is

$$\begin{bmatrix} 6xy & 3x^2 - 1 \\ 3x^2 - 1 & 0 \end{bmatrix}$$

x	y	D	f_{xx}	Type	Value
-1	0	-4	0	saddle	0
0	0	-1	0	saddle	0
1	0	-4	0	saddle	0

Problem 5) (10 points)

Use the Lagrange method to solve the problem to minimize

$$f(x, y) = x^2 + y^2 + xy$$

under the constraint $g(x, y) = 3x + 4y = 26$.

Solution:

The Lagrange equations are

$$\begin{aligned} 2x + y &= \lambda 3 \\ x + 2y &= \lambda 4 \\ 3x + 4y &= 26 \end{aligned}$$

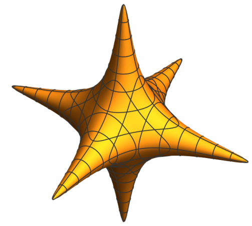
The solution is $(x, y) = (2, 5)$.

Problem 6) (10 points)

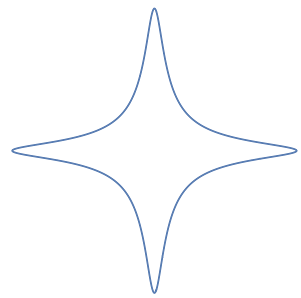
a) (5 points) When we look at the contour surface

$$f(x, y, z) = 10x^2y^2+10y^2z^2+10x^2z^2+(x^2+y^2+z^2) = 25 ,$$

we see some sharp spikes near the coordinate axes. To find out whether there is a singularity there, we compute the **tangent plane** at the point $(x_0, y_0, z_0) = (5, 0, 0)$.



b) (5 points) To understand this better, we cut up the surface at $z = 0$ and have a level curve $g(x, y) = 10x^2y^2 + x^2 + y^2 = 25$. Now find the **tangent line** $ax + by = d$ to this curve at the point $(x_0, y_0) = (5, 0)$.



Solution:

a) Compute the gradient at the point $(5, 0, 0)$ which is $[10, 0, 0]$. The equation is $10x = d$. Plug in the point to get $d = 50$. We have $x = 5$. b) Compute the gradient at the point $(5, 0)$ which is $[10, 0]$. The equation is $10x = d$. Plug in the point $(5, 0)$ to get $10x = 50$. So, $x = 5$.

Problem 7) (10 points)

On the last winter trip to **Martha's Vineyard**, "Olli Rocky Docky" and family took a steam boat. Olli uses a penny pressing machine there to deform a penny into a surface

$$\vec{r}(u,v) = [uv, u-v, u+v]$$

with parameters satisfying $0 \leq u^2+v^2 \leq 9$. Find the **surface area** of the surface. (Only one side of the penny of course).

Solution:

$r_u = [v,1,1], r_v = [u,-1,1]$. The cross product is $[2, u-v, -u-v]$. Its length is $|r_u \times r_v| = \sqrt{4+2u^2+2v^2}$. Now use polar coordinates $2\pi \int_0^3 \sqrt{4+2r^2} r \, dr$. This gives $\frac{2\pi}{3}(11\sqrt{22}-4) = (\pi/3)(22^{3/2}-8)$.

Problem 8) (10 points)

For more than a decade, there has been a "gold rush" in **Madagascar**. It is not gold but **sapphires** which attract the miners. Assume $f(x,y,z)$ is the probability to find a blue saphire. You know $f_z(1,2,3) = 1$ and measure

$$D_{\vec{v}} f(1,2,3) = 11/\sqrt{3}$$

if you go into the direction $\vec{v} = [1,1,1]/\sqrt{3}$. You also experience

$$D_{\vec{w}} f(1,2,3) = 7$$

for $\vec{w} = [3,4,0]/5$. To see how your luck changes when digging into the direction $\vec{u} = [1,-1,1]/\sqrt{3}$, compute the directional derivative $D_{\vec{u}} f(1,2,3)$.

Solution:

We introduce the gradient $\nabla f = [a,b,c]$. Then write down the equations $c = 1, a+b+c = 11, 3a+4w = 35$ which gives. $a = 5, b = 5, c = 1$ The result is $[a,b,c].[1,-1,1]/\sqrt{3} = 1/\sqrt{3}$.

Problem 9) (10 points)

a) (5 points) Evaluate the double integral

$$\int_1^e \int_{\log(x)}^1 \frac{y}{e^y - 1} \, dydx \ ,$$

where log is the natural log as usual.

b) (5 points) Evaluate the double integral

$$\iint_G \log(x^2 + y^2) \, dxdy \ ,$$

where G is region given by

$$\{4 \leq x^2 + y^2 \leq 9\} \ .$$

P.S. as a grown-up, you also know that integration by parts gives $\int \log(x) \, dx = x \log(x) - x$.

Solution:

a) Change the order of integration. We end up with the integral

$$\int_0^1 \int_1^{e^y} \frac{y}{e^y - 1} \, dxdy \ .$$

Now, the inner integral is no problem and gives y. The result is $\int_0^1 y \ dy = \boxed{1/2}$. b) (5 points) Use polar coordinates. $2\pi \int_2^3 \log(r^2) r \ dr = -5\pi + 9\pi \log(9) - 4\pi \log(4)$ For evaulating the integral substitute $r^2 = u$ and use the hint.