

7/25/2024 SECOND HOURLY Practice 5 Maths 21a, O.Knill, Summer 2024

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points). No justifications needed.

- 1) ☒ T ☐ F The relation $f(x, y) = \sin(x + \sin(y)) = 0$ defines a function $y = y(x)$ near $x = 0$. Its derivative is $y'(x) = -f_x(0, 0)/f_y(0, 0) = -1$.

Solution:

Yes, this is implicit differentiation.

- 2) ☒ T ☐ F The point $(0, 0)$ is the only critical point of $f(x, y) = x^9 - y^3$.

Solution:

There is only one critical point $(0, 0)$

- 3) ☐ T ☒ F The gradient of $f(x, y, z) = x^2 + y^2 - z$ at $(1, 2, 5)$ is perpendicular to $\vec{r}_u \times \vec{r}_v$, if $\vec{r}(u, v) = [v \cos(u), v \sin(u), v^2]$ is the parametrization of $f(x, y, z) = 0$.

Solution:

It is perpendicular to $\vec{r}_u$ or $\vec{r}_v$, but **parallel** to the gradient.

- 4) ☐ T ☒ F The surface area of a surface parametrized by $\vec{r}(u, v)$ over a domain R in the uv -plane is given by the integral $|\int_R \vec{r}_u \times \vec{r}_v \, du dv|$.

Solution:

We need $|\vec{r}_u \times \vec{r}_v|$.

- 5) ☐ T ☒ F If $f_{xy}(x, y) = 1$ everywhere, then f can only have saddle points.

Solution:

Take for example $x^2/2 + y^2/2 + xy = (x + y)^2/2$

- 6) ☐ T ☒ F The surface area of a cone of height 2 and radius 1 is the same than the surface area of the unit sphere.

Solution:

Archimedes discovered such relations, but not this one.

- 7) ☒ T ☐ F The function $f(x, y) = x + x^2 - 7y^2$ has a global maximum under the constraint $x = 0$.

Solution:

Just put $x = 0$ and see $-7y^2$.

- 8) ☒ T ☐ F Assume $(1, 1)$ is a local maximum of g and $(1, 1)$ is a local minimum of a function $f(x, y)$, then $(1, 1)$ is a solution of the Lagrange equations.

Solution:

Both gradients are zero.

- 9) ☐ T ☒ F If $(0, 0)$ is a saddle point for f with $D < 0$, then the directional derivative $D_{\vec{v}}f(0, 0)$ takes positive and negative values if $\vec{v}$ ranges over all unit vectors.

Solution:

All directional derivatives are zero.

- 10) ☒ T ☐ F If $(0, 0)$ is a saddle point for f , then $f_{xx}(0, 0)$ and $f_{yy}(0, 0)$ can have the same signs.

Solution:

Take $x^2 + y^2 + 100xy$ for example. It has a saddle point at $(0, 0)$.

- 11) ☒ T ☐ F For $\vec{u} = [1, 0]$ and $\vec{v} = [0, 1]$ then the discriminant D satisfies $D = (D_{\vec{u}}D_{\vec{u}}f)(D_{\vec{v}}D_{\vec{v}}f) - (D_{\vec{u}}D_{\vec{v}}f)^2$, where $D_{\vec{u}}, D_{\vec{v}}$ are directional derivatives.

Solution:

This is the definition, if noticing that $f_u = D_{\vec{u}}f$.

- 12)

T	F
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 If $\vec{r}(t)$ is a curve on the surface $f(x, y, z) = 1$, then the unit tangent vector $\vec{T}(\vec{r}(t))$ is perpendicular to $\nabla f(\vec{r}(t))$.

Solution:

Take the grand circle on the sphere. The normal vector is now perpendicular to the surface.

- 13)

T	F
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 The function $f(x, y) = 4x^3y^2 + \sin(\sin(x^{10})) + \arctan(y^8)$ solves the partial differential equation $f_{xyxyxy} = 0$.

Solution:

Use Clairaut

- 14)

T	F
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 If $f_x(x, y) = f_y(x, y)$ for all x, y , then $f_{xx}(x, y) = f_{yy}(y, x)$ for all (x, y) .

Solution:

Differentiate f_x with respect to y to get f_{xy} . Differentiate f_y with respect to x to get f_{yx} . By Clairaut these are the same.

- 15)

T	F
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 The double integral $\int_0^x \int_0^1 x^2 dx dy$ is a rational number.

Solution:

It is not a number, it is a function of x .

- 16)

T	F
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 As we have learned from “gifted”: $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

Solution:

This was from the movie clip gifted where the girl was given the wrong sign problem

- 17)

T	F
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 Using linearization we can estimate $\sqrt{101} \sim 10 + 1/20 = 10.05$.

Solution:

Yes, this is the linearization. The actual value is 10.04999

- 18) ☐ T ☒ F If $f(x, t)$ solves the heat equation, then $f(x, -t)$ solves the wave equation.

Solution:

The heat equation can not be put backwards. Take $f(x, t) = 2t - x^2$. It solves the heat equation but $-2t - x^2$ does not.

- 19) ☐ T ☒ F There is a function f such that $D_{\vec{u}}f(0, 0)$ is positive for all unit vectors $\vec{u}$.

Solution:

If the function is positive in one direction, then it is negative in the opposite direction.

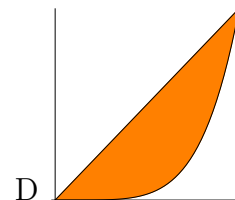
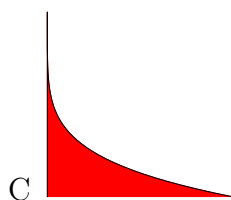
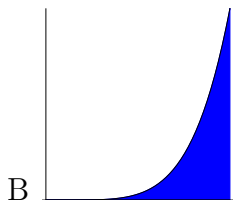
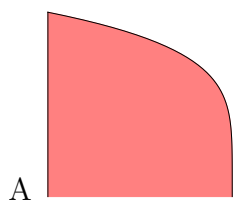
- 20) ☐ T ☒ F The differential equation $f_{xx} - f_x = f_t$ is called the wave equation.

Solution:

False

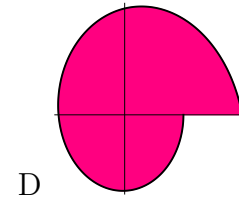
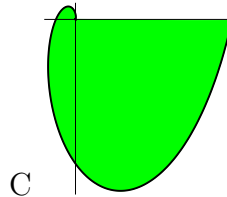
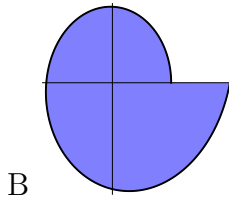
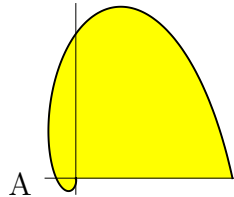
Problem 2) (10 points) No justifications are needed

- a) (4 points) Match the following regions with their area computation. There is an exact match.



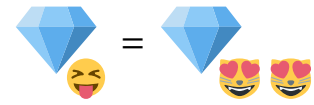
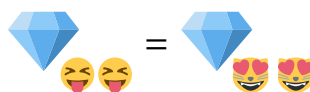
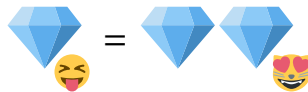
Enter A-D	Area Integral
	$\int_0^1 \int_{x^5}^x 1 \, dydx$
	$\int_0^1 \int_0^{x^5} 1 \, dydx$
	$\int_0^1 \int_0^{1-x^5} 1 \, dydx$
	$\int_0^1 \int_0^{1-y^5} 1 \, dx dy$

b) (4 points) Match the following regions with their area computation. There is an exact match.



Enter A-D	Area Integral
	$\int_0^{2\pi} \int_0^{\theta^3} r \, dr d\theta$
	$\int_0^{2\pi} \int_0^{(2\pi)^3 - \theta^3} r \, dr d\theta$
	$\int_0^{2\pi} \int_0^{1+\theta^3} r \, dr d\theta$
	$\int_0^{2\pi} \int_0^{1+(2\pi)^3 - \theta^3} r \, dr d\theta$

c) (2 points) Name all the three PDE's. The symbols are replaced by Emojis. Each Emoji is either a variable or a function. Remember that **who ever eats hamburgers during heat waves is transported to La la place!**



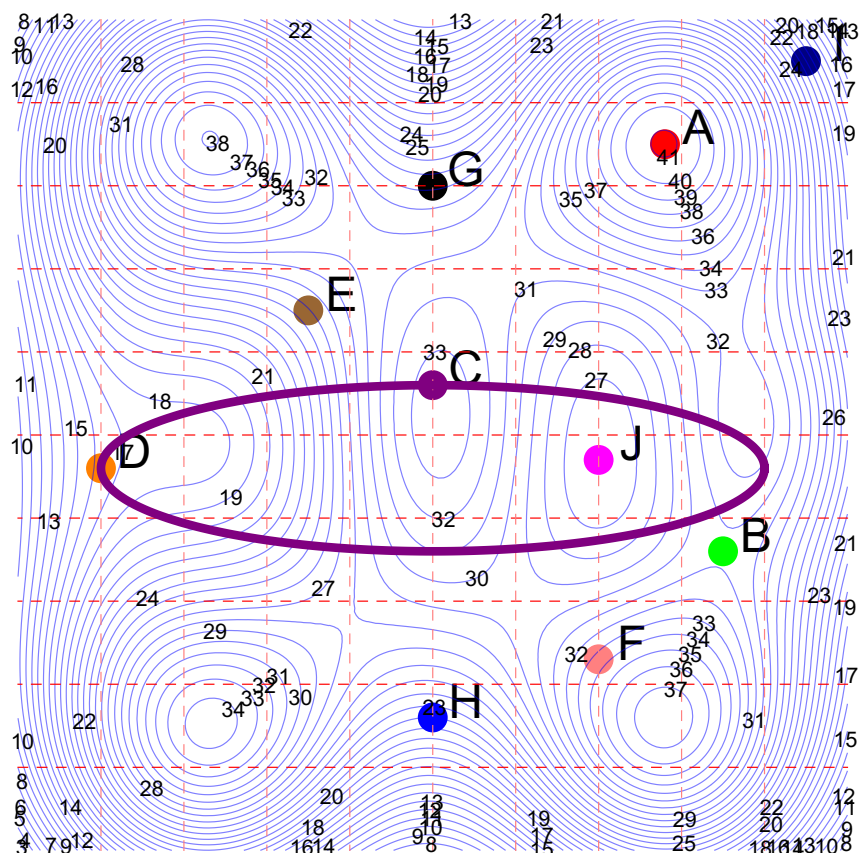
Solution:

- a) DBAC
b) CABD
c) Burgers, Wave, Heat.

Problem 3) (10 points) (No justifications are needed.)

(10 points) For each question, there is exactly one of the 10 points A-J which fits the bill. Define $\vec{v} = [1, 1]/\sqrt{2}$, $\vec{w} = [1, -1]/\sqrt{2}$. An ellipse $g(x, y) = c$ is a constraint for two questions.

	Enter your choice of A-J
A point with $f_x = 0, f_y < 0, f_{xx} > 0$	
A saddle point	
A maximum under the constraint $g = c$	
A minimum under the constraint $g = c$	
$ f_{xx} $ is largest among A-J	
$ \nabla f $ largest among A-J	
A local max of f	
A local min of f	
A point where $D_{\vec{v}}f > 0, D_{\vec{w}}f = 0$	
A point where $D_{\vec{w}}f > 0, D_{\vec{v}}f = 0$	



Redemption! Here is a chance to get one or the other points back. The answer must be perfect!

(1 point) Formulate the chain rule $\frac{d}{dt}f(\vec{r}(t)) = \dots$

(1 point) What is the direction of maximal ascent for f at a non-critical point (x_0, y_0) ?

(1 point) What does Fubini's theorem tell?

Solution:

G,B,C;D;H,I,A,J,E,F

Problem 4) (10 points)

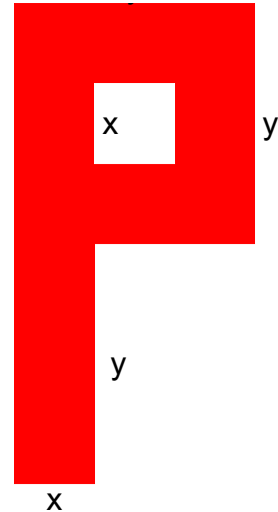
What is the optimal geometry for the **letter P**? In order to find out, we maximize a weighted area of the letter under the constraint that the boundary length is fixed. The area scales the bottom part 4 times.

$$f(x, y) = 4xy + y^2 - x^2 .$$

The total boundary length is

$$g(x, y) = 4x + 6y = 58 .$$

Use Lagrange to find the maximal f value under this constraint.



Solution:

The Lagrange equations give one critical point $x = 4, y = 7$.

Problem 5) (10 points)

Meet

$$f(x, y) = x^3 - 2y^2 - 12xy$$

It is one of the millions overlooked functions in math. It is called “f” and it is sad, as nobody has looked at it yet and recognized its talent. This is the chance for 15 minutes of fame for “f”.

a) (8 points) Find the maximum, minima or saddle points of “f” using the second derivative test.

b) (2 points) Does “f” have a global maximum or global minimum? Explain.

Give it up for “f”! We already reserved a star at the Hollywood hall of fame.



Solution:

- a) There are two critical points: $(-12, 36)$ is a local max and $(0, 0)$ is a saddle.
b) There is no global maximum.

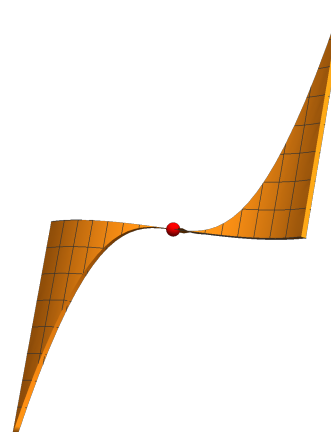
Problem 6) (10 points)

a) (5 points) To compute the surface area of a **propeller**, compute the surface area of

$$\vec{r}(u, v) = \begin{bmatrix} v \\ \frac{u^3}{3} + v \\ 2u \end{bmatrix}$$

with $0 \leq v \leq 1$ and $v^{1/3} \leq u \leq 1$. (The propeller will have twice the area).

b) (5 points) Now solve the integral and don't forget to use what you have learned about double integrals.



Solution:

Set up the integral. $|r_u \times r_v| = \sqrt{8 + u^4}$,

$$\int_0^1 \int_0^{v^{1/3}} \sqrt{u^4 + 8} \, du dv .$$

Change the order of integration to get

$$\int_0^1 \int_0^{u^3} \sqrt{u^4 + 8} \, dv du .$$

Now integrate

$$\int_0^1 u^3 \sqrt{u^4 + 8} \, du$$

to get $\boxed{(9/2 - 16\sqrt{2}/6)}$. Multiply by to get $\boxed{(9 - 16\sqrt{2}/3)}$.

Problem 7) (10 points)

Knowing tangent planes is important in **ray tracing**, where the computer bounces light around in the scene.

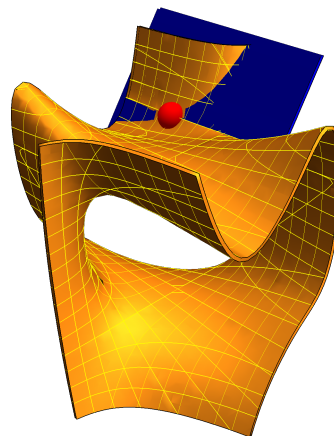
a) (5 points) Find the tangent plane to the surface

$$xyz - x^5y + z^2 = 1$$

at the point $(1, 1, 1)$.

b) (5 points) Near $(x, y) = (1, 1)$, we can write $z = g(x, y)$. Find the gradient

$$\nabla g(1, 1) = \begin{bmatrix} g_x(1, 1) \\ g_y(1, 1) \end{bmatrix} .$$



Solution:

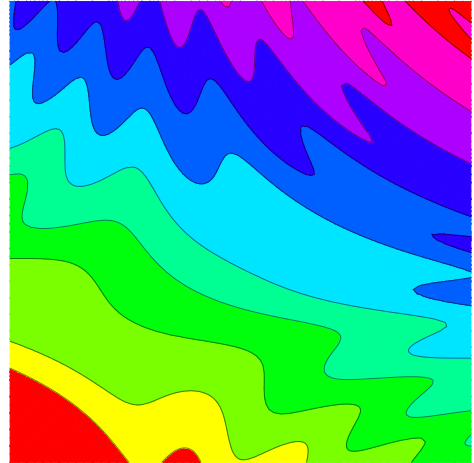
a) $\nabla f = [yz - 5x^4y, xz - x^5, 2z + xy]$ $\nabla f(1, 1, 1) = [-4, 0, 3]$.

The equation is $\boxed{-4x + 3z = -1}$.

b) $g_x = -f_x/f_z = 4/3, g_y = -f_y/f_z = 0$. So $\nabla g = [4/3, 0]$. There is an solution if one takes $(1, 1, -2)$.

Problem 8) (10 points)

As an **AI experiment**, we ask our computer to draw some art. Here is what it came up with: it painted the contour map of a function f and called it “**Shaken, but not stirred, 2019**”. What strange ideas computers come up with! It must have watched some of the movies which were on the harddrive.



a) (5 points) Find the linearization $L(x, y)$ of

$$f(x, y) = 1 + x + y^2 + \sin(xy)$$

at $(0, 0)$.

b) (5 points) Estimate $f(0.01, 0.03)$ using linear approximation. The function is the same as the function in a).

Solution:

- a) The gradient of f is $\nabla f(x, y) = [1 + y \cos(xy), 2y + x \cos(xy)]$ which is $[a, b] = [1, 0]$ at $(0, 0)$. The linearization is $L(x, y) = 1 + x$.
- b) The linear estimation is $L(0.01, 0.03) = 1 + 0.01 = 1.01$.

Problem 9) (10 points)

In order to solve the following problems, remember all the **advise** we have given about solving double integrals.

a) (5 points) Find

$$\iint_G \frac{5y}{x^2 + y^2} dx dy ,$$

where G is the region $1 \leq x^2 + y^2 \leq 25$, $y > 0$.

b) (5 points) Solve the following double integral

$$\int_1^2 \int_{e^y}^{e^2} \frac{x-1}{\log(x)-1} dx dy .$$



Solution:

a) Use polar coordinates and write the integral as $\int_0^\pi \int_1^5 5 \sin(\theta) dr d\theta = 40$.

b) Make a picture and change the order of integration:

$$\int_e^{e^2} \int_1^{\log(x)} \frac{x-1}{\log(x)-1} dy dx$$

which is $\int_e^{e^2} x - 1 dx = (x^2/2 - x)|_e^{e^2} = e^4/2 - 3e^2/2 + e$.