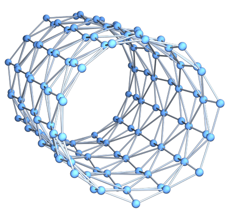
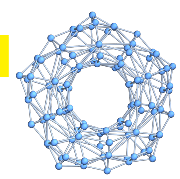
A glimpse onto Math S 21a taught in 2050

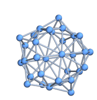
Calculus is the theory of **change** and **accumulation**. It also works in the discrete. Already the founders of calculus illustrated calculus ideas with discrete notions and thought about infinitesimals. We give here a **dictionary**, where 24 topics of our summer course are compared to notions in **discrete calculus**, a part of combinatorics. A nice finite geometric frame-work can be a **simplicial complex**, which is a finite collection of finite non-empty sets closed under taking non-empty subsets. If these sets are the nodes and two sets are connected by edges if one is contained in the other, one obtains a **graph** to visualize the space. A bit more general than simplicial complexes is a finite set of sets with a notion of dimension and exterior derivative. One calls this a **delta set**. There have been various attempts to combine the **quantum world** with the **geometry of gravity**. Example approaches are **causal sets**, **causal dynamical triangulation** or **loop quantum gravity**. Which mathematical structures matter in a **quantum gravity**? Why do we see the structures of the standard model in particle physics? Can one compute numbers like the **fine structure constant**? It is likely that a future language describing fundamental processes in nature will be some form of **calculus** meaning having a notion of derivative and a compatible notion of integration. It is important to note that a successful theory will have to be able to do things well **on a micro scale**, **on the meso scale of our lives** as well as **on the macro scale**: it has to describe the features of atoms and molecules to do chemistry, predict the motion of a planetary system for millions of years and also describe correctly large scale physics like for example what happens if two black hole merge. It also has to predict outcomes, where current models do not even suggest anything, like what happens with the singularity just when a black hole has completely evaporated. [1]

Chapter 1. Geometry and Space

space	a geometry where the smallest spheres are 2-spheres
surface	a geometry for the smallest spheres are 1-spheres
distance	length of shortest geodesics between two points
a sphere	a space which when punctured becomes contractible







Chapter 2: Curves and Surfaces

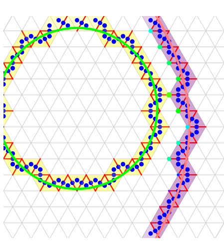
parametrization	an embedding of a smaller dimensional space
contour	set of elements on which $f - c$ changes sign
arc length	number of edges
surface area	number of triangles

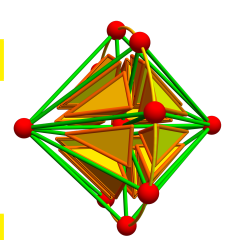
Chapter 3: Linearization and gradient

gradient	a 1-form: a function on edges f(a,b) = f(b)-f(a)
curl	a 2-form: a function on triangles f(a,b,c) = f(a,b)+f(b,c)+f(c,a)
div	a 3-form a function on tetrahedra adding up the boundary
Laplacian	Kirchhoff Laplacian

Chapter 4: Extrema and Double integrals

critical point	point a where $S = \{f(x) < f(a)\}$ is not contractible
discriminant	the index $1 - \chi(S_f^-)$ at a critical point
double integral	sum of the function values over triangles
surface area	the number of triangles





Chapter 5: Integrals of fields

scalar integral	sum up 0-form on a discrete set of points
line integral	sum up 1-form along a curve
flux integral	sum of 2-form over a surface
triple integral	sum of 3-form over a solid

Chapter 6: Integral theorems

FTLI	defined for a curve, a collection of edges
Green	defined on a surface, a collection of triangles
Stokes	surface with boundary, defined for collection of triangles
Gauss	region with boundary, defined for a collection of tetrahedra

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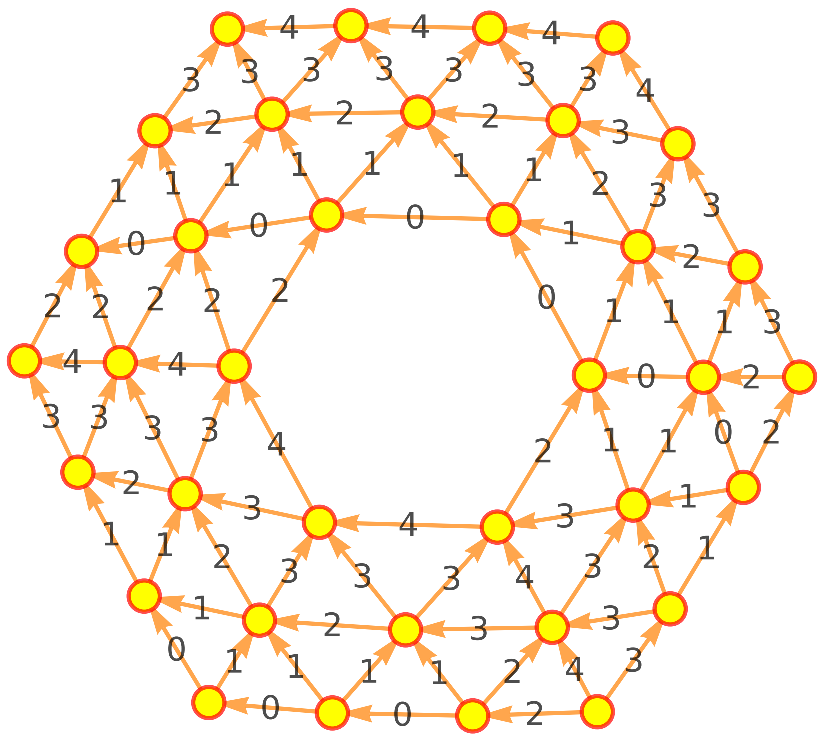
[1] This question was once posed by the physicist Gerard 't Hooft. It involves both gravity and quantum mechanics. Both of these pillars use multi-variable calculus and linear algebra on steroids in the form of differential geometry and functional analysis.

An example. (see final exam of Math S 21a of Summer 2025.

A **discrete surface** is a graph in which the nearest neighbors of each node form either a cyclic graph or a path graph. Interior points have entire circles around them while boundary points only have discrete intervals as nearest neighbors. The picture below shows a surface. It is a discrete cylinder or annulus. It has two boundary curves, both of which are circles. The orientation given on edges plays the role of orientations we use in the continuum when parametrizing curves. The function $\vec{F}$ on edges is a **vector field**. It describes how much is transported by the field from a node to each neighboring node. Gradient field ∇f of a scalar function f form a class of vector fields $\nabla f((a,b)) = f(b) - f(a)$, functions on edges. If a path goes from a node a to a node b then the line integral $\int_C F dr$ is the value $F((a,b))$. If the path goes from b to a, the line integral is $-F(a,b))$. The curl of a vector field is a function on triangles (abc) (complete subgraphs with 3 nodes). It is $F((a,b)) + F(b,c) + F(c,a)$, the line integral along the boundary of the triangle. You can easily check that if $F = \nabla f$ and C is a curve going from a to b, then $\int_C F dr = f(b) - f(a)$. For a closed curve, we have $\int_C \nabla f \; dr = 0$. Applied to a triangle, we see that the curl of a gradient field is zero for every triangle. For a surface S with boundary C oriented so that surface is to the left, we have $\int_S \text{curl}(F) \; dS = \int_C F \; dr$. This is Stokes theorem.



An exercise:

Use Stokes theorem to compute the flux of the curl of $\vec{F}$ through S.