**8/7/2025 FINAL EXAM** **Maths 21a, O.Knill, Summer 2025**

**"I affirm my awareness of the standards of the Harvard College Honor Code."**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

| | | |
|---|---|---|
| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 10 | | 10 |
| 11 | | 10 |
| 12 | | 10 |
| 13 | | 10 |
| Total: | | 140 |

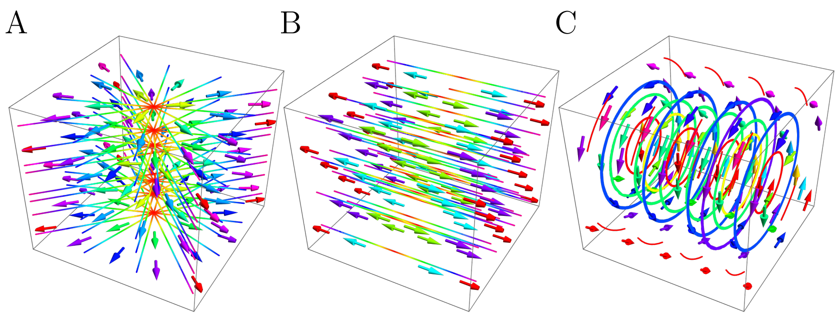
Problem 1) (20 points) No justifications are needed.

| | | | |
|---|---|---|---|
| 1) | T | F | The vector field $\vec{G} = [0, xz, 0]$ has the property that its curl is $[x, 0, -z]$. |
| 2) | T | F | Every vector field is the curl of an other vector field. |
| 3) | T | F | A solid torus is simply connected. |
| 4) | T | F | The partial differential equation $\text{div}(\vec{F}) = 4\pi\rho$ describes gravity: it allows to derive the Newton law of gravity. |
| 5) | T | F | The curve $x^2 + y^2 = 9, z = \sin(x)$ is parametrized by $\vec{r}(t) = [3\cos(t), 3\sin(t), \sin(3\cos(t))]$. |
| 6) | T | F | The vector field $\text{curl}(\text{curl}(\vec{F}))$ is constant 0. |
| 7) | T | F | The curvature of the curve $\vec{r}(t) = [0, 5\sin(t), 5\cos(t)]$ is $1/5$ everywhere. |
| 8) | T | F | Any function that satisfies the transport equation $f_t = f_x$ and heat equation $f_t = f_{xx}$ must be linear in $x$. |
| 9) | T | F | The Maxwell equations in 3 dimensions consist of 4 equations. |
| 10) | T | F | The boundary of the cylinder $x^2+y^2 = 1, -1 \leq z \leq 1$ consists of two closed circles. |
| 11) | T | F | The curl of the divergence of a vector field is defined. |
| 12) | T | F | We can the divergence theorem to compute volumes of solids. |
| 13) | T | F | The quadric $x^2 - y^2 - (z-1)^2 = 4$ is a one-sheeted hyperboloid. |
| 14) | T | F | If $\vec{r}(u, v)$ parametrizes a surface, then $\vec{r}_u$ is tangent to the surface. |
| 15) | T | F | The gradient of the gradient is defined. |
| 16) | T | F | If $\vec{r}(t)$ parametrizes a parabola, then the jerk of the curve is zero. |
| 17) | T | F | The boundary of a closed curve is empty. |
| 18) | T | F | The vector projection of $\vec{F} = [P, Q, R]$ onto $[1, 1, 1]$ is the divergence of $\vec{F}$. |
| 19) | T | F | Both the curvature and the torsion of a line is constant zero. |
| 20) | T | F | The vector $[-2, -2, 1]$ is perpendicular to the surface $g(x, y, z) = z - x^2 - y^2 = -1$ at the point $(1, 1, 1)$. |

| Problem 2) (10 points) No justifications are necessary. |
|---|

a) (2 points) The figures display vector fields in space. There is an exact match.

| Field | A-C |
|---|---|
| $\vec{F} = [x, 0, 0]$ | |
| $\vec{F} = [0, -z, y]$ | |
| $\vec{F} = [x, y, 0]$ | |




b) (2 points) Fill in the dimension numbers $m$ and $n$ into the table.

| The object | maps | $\mathbb{R}^m$ | to | $\mathbb{R}^n$ |
|---|---|---|---|---|
| $f(x, y)$ | maps | | to | |
| $\vec{F}(x, y, z) = [P, Q, R]$ | maps | | to | |
| $\vec{r}(t) = [x(t), y(t)]$ | maps | | to | |
| $\vec{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$ | maps | | to | |

c) (4 points) Match the triple integrals with solids.

| Enter A-D | 3D integral computing volume |
|---|---|
| | $\int_{\pi/2}^{3\pi/2} \int_{\pi/4}^{3\pi/4} \int_0^2 \rho^2 \sin(\phi)\ d\rho d\phi d\theta$ |
| | $\int_{\pi/2}^{3\pi/2} \int_{\pi/4}^{3\pi/4} \int_1^2 \rho^2 \sin(\phi)\ d\rho d\phi d\theta$ |
| | $\int_{\pi/2}^{3\pi/2} \int_0^{\pi/4} \int_1^2 \rho^2 \sin(\phi)\ d\rho d\phi d\theta$ |
| | $\int_{\pi/2}^{3\pi/2} \int_{\pi/6}^{\pi/4} \int_0^2 \rho^2 \sin(\phi)\ d\rho d\phi d\theta$ |

A B C D

e) (2 points) Write down the names two partial differential equations involving functions of two variables for which both variables are differentiated twice.

| | |
|---|---|
| | |

| Problem 3) (10 points) | No justifications are needed. |
|---|---|

Choose from the following words to complete the table below. There are 13 entries, every wrong entry gives one point off. If more than 10 should be wrong, the score would be zero. Oliver would be so embarrassed to see this however.

"Pythagoras theorem", "Al Khashi cos-formula", "double angle formula" "distance formula", "Clairaut test", "curvature", "jerk", "torsion", "crack", "snap", "curl", "second derivative test", "Lagrange equations", "arc length formula", "surface area formula", "chain rule", "volume of parallel epiped", "area of parallelogram", "partial differential equation", "Stokes theorem", "Cauchy-Schwarz inequality "Fubini Theorem", "line integral", "flux integral", "fundamental theorem of line integrals", "vector projection", "scalar projection", "partial derivative", "unit tangent vector", "normal vector", "binormal vector", "Green's theorem".

| Formula | Formula or rule or theorem |
|---|---|
| $\cos^2(\theta) = \frac{\cos(2\theta)+1}{2}$ | |
| $\vec{r}\,'(t)/\lvert\vec{r}\,'(t)\rvert$ | |
| $\lvert\vec{PQ} \cdot (\vec{v} \times \vec{w})\rvert/\lvert\vec{v} \times \vec{w}\rvert$ | |
| $\lvert\vec{r}\,'(t) \cdot (\vec{r}\,''(t) \times \vec{r}\,'''(t))\rvert/\lvert\vec{r}\,'(t) \times \vec{r}\,''(t)\rvert$ | |
| $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ | |
| $\int_a^b \lvert\vec{r}\,'(t)\rvert \; dt$ | |
| $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \; dt$ | |
| $\mathrm{div}(\mathrm{curl}(\vec{F})) = 0$ | |
| $\lvert\vec{v} - \vec{w}\rvert^2 = \lvert\vec{w}\rvert^2 + \lvert\vec{v}\rvert^2 - 2\lvert\vec{v}\rvert\lvert\vec{w}\rvert \cos(\theta)$ | |
| $\vec{T} \times \vec{N}$ | |
| $\lvert\vec{r}\,'(t) \times \vec{r}\,''(t)\rvert/\lvert\vec{r}\,'(t)\rvert^3$ | |
| $\lvert\vec{v} \cdot \vec{w}\rvert \leq \lvert\vec{v}\rvert\lvert\vec{w}\rvert$ | |
| $\lvert\vec{v} \times \vec{w}\rvert$ | |

Problem 4) (10 points)

In the following problem, do not bother with integral theorems. We want you just to compute the integrals directly. This is **"action directe"**, the direct route.

Wolfgang Guellich was the first who climbed the legendary "Action Direct" boulder route in Germany.

Look at the **helix**

$$C : \vec{r}(t) = \begin{bmatrix} \cos(10t) \\ \sin(10t) \\ t \end{bmatrix}, 0 \leq t \leq 2\pi$$

the **helixoid**

$$S : \vec{r}(s,t) = \begin{bmatrix} s\cos(10t) \\ s\sin(10t) \\ t \end{bmatrix}, 0 \leq t \leq 2\pi, 0 \leq s \leq 1$$

and the vector field

$$\vec{F}(x,y,z) = \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}.$$

a) (2 points) Compute the arc length $\int_C dr$ of the curve.

b) (2 points) Compute the line integral $\int_C \vec{F} \cdot \vec{dr}$.

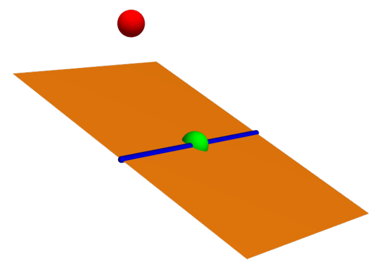
c) (2 points) Write down the surface area integral $\int\int_S dS$ and simplify until a single integral is reached. We have done that integral in class, but you do not have to evaluate that.

d) (2 points) Compute the flux integral $\int\int_S \vec{F} \cdot \vec{dS}$.

e) (2 points) Which of the integrals a),b),c),d) do **not** depend on the orientation and parametrization of the curve?

Problem 5) (10 points)

Let $P = (1, 2, 1)$ and $Q = (1, 1, 1)$ and $\vec{v} = [1, 0, 1]$ and $\vec{w} = [1, 1, 0]$. Define the line $L : \vec{r}(t) = Q + t\vec{v}$ and the plane $S : \vec{r}(t, s) = Q + t\vec{v} + s\vec{w}$. The point Q, the line L and the plane S are **stratified** meaning $Q$ is part of $L$ and $L$ is part of $S$. A general principle assures that $d(P, Q) \geq d(P, L) \geq d(P, S)$ in such a case. Verify this in the current situation:



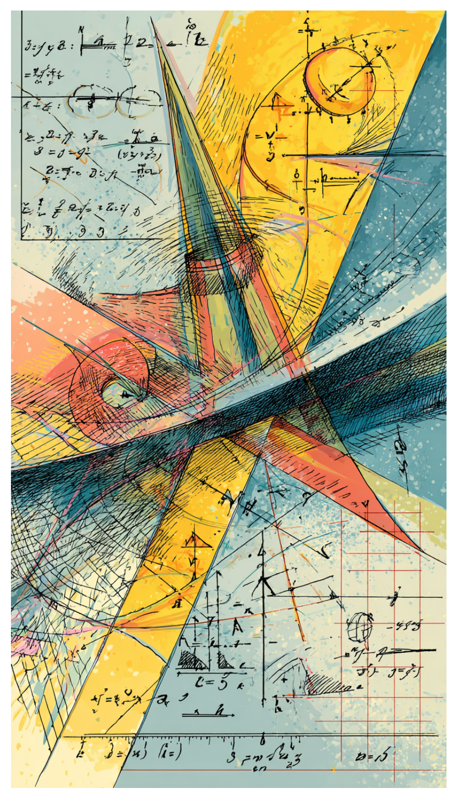
a) (2 points) Compute $d(P, Q)$.

b) (4 points) Compute $d(P, L)$.

c) (4 points) Compute $d(P, S)$.

Problem 6) (10 points)

The following problem is a winner of the first **"Math exam poetry competition."**



Picture AI generated

a) (2 points) You know that the tangent plane to an unknown surface $f(x, y, z) = 10$ at the point $P = (1, 2, 3)$ is $4x + 5y + 6z = 32$. You are also told that $f_x(1, 2, 3) = 4$. Estimate $f(1.01, 2.001, 3.0001)$.

b) (2 points) What is the directional derivative $D_{\vec{v}}f$ of $f$ at $(1, 2, 3)$ in the direction $\vec{v} = [1, 1, 1]/\sqrt{3}$?

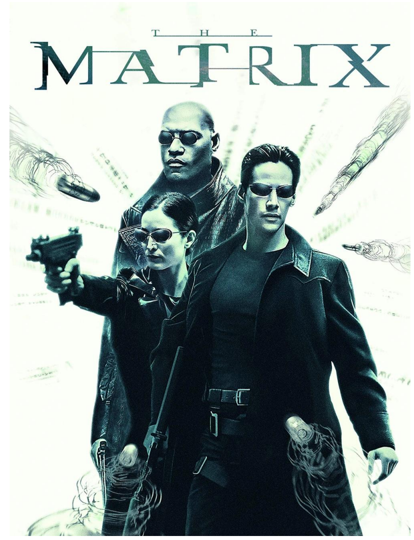
c) (2 points) Assume the level surface $f(x, y, z) = 10$ can be written as a graph $z = g(x, y)$ with $3 = g(1, 2)$. Find $g_x(1, 2)$ and $g_y(1, 2)$. [1]

d) (2 points) Estimate $g(0.9, 1.99)$.

e) (2 points) Find the directional derivative $D_{\vec{w}}g$ of $g$ at $(1, 2)$, if $\vec{w} = [1, 1]/\sqrt{2}$.

Problem 7) (10 points)

Problem b) below should trigger some **“déja vue”** feeling. If you have seen the ”matrix” you know the sensation. It is a glitch in the matrix.




a) (5 points) Compute the triple integral $\iiint_E f \ dV$ where

$$f(x,y,z) = x^2$$

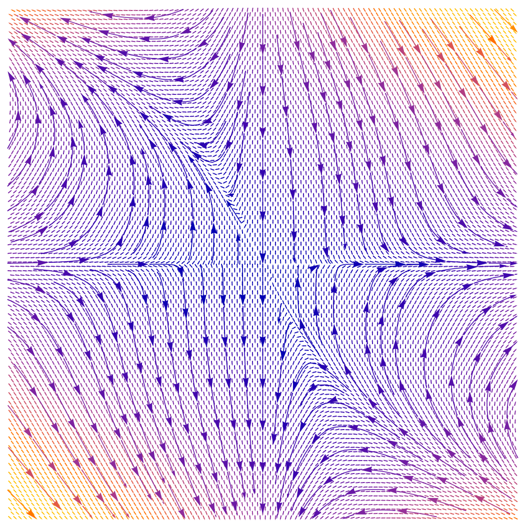
and where the solid is

$$E = \{9 < x^2 + y^2 \leq 25, y \geq 0, 0 \leq z \leq 1\} .$$

b) (5 points) Find the surface area of the surface $S$ parametrized as $\vec{r}(x,y) = [x,y,0]$, where the parameter domain is given by

$$R = \{0 \leq y \leq 1, 0 \leq x \leq \arccos y\} .$$

Problem 8) (10 points)

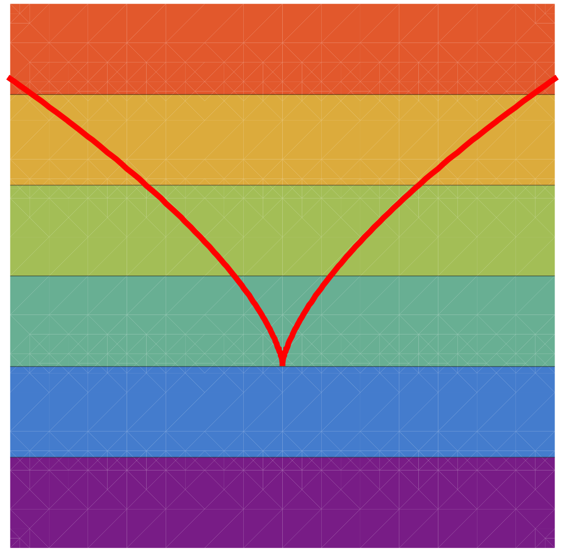


The divergence of a vector field $\vec{F} = [P,Q]$ in two dimensions is the scalar function $\text{div}(\vec{F}) = P_x + Q_y$. Use the **second derivative test** to classify all the critical points of the scalar function $f = \text{div}(\vec{F})$, where

$$\vec{F} = \left[ \begin{array}{c} \frac{x^4}{4} + xy^3 \\ -3xy - \frac{3y^2}{2} \end{array} \right] .$$

Make a list of all critical points of $f$ and indicate whether they are maxima, minima or saddle points.

Problem 9) (10 points)



We try to extremize the function $f(x,y) = y$ under the contraint $g(x,y) = x^2 - y^3 = 0$ using the method of Lagrange.

a) (4 points) Write down the Lagrange equations.

b) (4 points) Verify that there is no solution to the Lagrange equations.

c) (2 points) The Lagrange method did not "see" $(0,0)$, the minimum of $f$ on $\{g(x,y) = 0\}$. We discussed this phenomenon in class. Explain why Lagrange can fail to find the minimum.

Problem 10) (10 points)

Motivated by the "Soaky mountain" water park, we build a water slide

$$\vec{r}(t) = \begin{bmatrix} \sin(2\pi t) + \sin(10\pi t) \\ \cos(2\pi t) + \cos(10\pi t) \\ 15 - 15\sin((\pi/2)t) \end{bmatrix}, 0 \leq t \leq 1$$

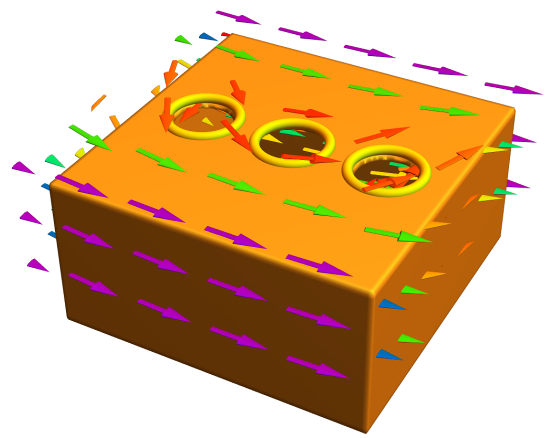
The combined wind, water and gravity force field on a rider is the vector field

$$\vec{F}(x,y,z) = \begin{bmatrix} \cos(x) + 3xy^2 \\ 10y^4 + 3x^2y \\ -z \end{bmatrix}.$$

Find the line integral $\int_0^1 \vec{F} \cdot d\vec{r}$. It is the energy you gained by riding the slide.

Problem 11) (10 points)

The hull $S$ of a **tank** is the boundary of the cuboid $-10 \le x \le 10, -10 \le y \le 10, -10 \le z \le 0$ intersected with $(x-6)^2 + y^2 + z^2 \ge 4$ and $x^2 + y^2 + z^2 \ge 4$ and $(x+6)^2 + y^2 + z^2 \ge 4$. The boundary of $S$ is made of 3 circles of radius 2 in the $xy$-plane. Find the flux of curl of the vector field $\vec{F} = \begin{bmatrix} -y + zx + zy \\ x + zx^3 + z \\ z^5 + \sin(z) + y^5 \end{bmatrix}$ through the surface $S$, assuming that $S$ is oriented outwards.

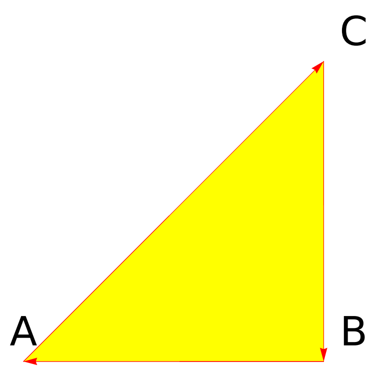


Problem 12) (10 points)

Evaluate the line integral of the vector field

$$\vec{F}(x,y) = \begin{bmatrix} x^6 - 4y^2 + e^{\sin(x)} \\ x^2 + y^7 \sin^2(15y) \end{bmatrix}$$

in the **clockwise direction** around the triangle G=ACB in the plane defined by the points $A = (0,0), B = (1,0)$ and $C = (1,1)$.

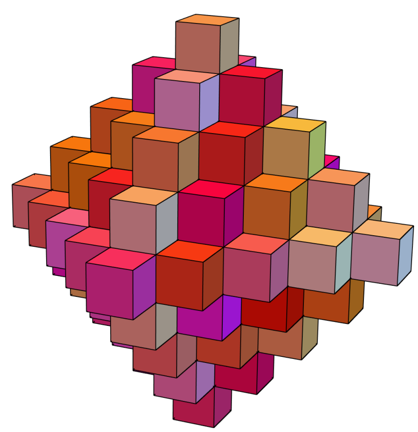



Problem 13) (10 points)

Find the flux of the vector field

$$\vec{F}(x,y,z) = \begin{bmatrix} 5x + z^2 \\ 8y - \sin(x^3) + e^z \\ -9z + \cos(x^2 + y^2 + 1) \end{bmatrix}$$

through the boundary $S$ of the solid $E$ displayed in the picture. The surface is oriented outwards as usual. It is an **art work** made of 129 cubes where each has side length 1.



Problem 14 *) (5 points) This quantum calculus problem can be ignored.

We only looked at this quantum calculus topic briefly in the last lecture of the 6th week. If you managed to understand this, it is an opportunity to regain some lost points (maximally 5). If you have decided to ignore this topic, do not worry and focus your energy to the actual exam problems 1-13. As mentioned in the intro lecture, this topic could be a glimpse into the future. Maybe, in 2055, Stokes theorem will be taught like this. By the way, the classical Stokes theorem had first appeared in writing in a multi-variable calculus exam given by Stokes. We have also seen that James Maxwell (from the Maxwell equations describing all electromagnetic phenomena) took that exam as a student and got the best score.

Here is the beginning of the story again: a discrete surface is given by a graph in which the neighbors of each node form either a cyclic graph or a path graph. The former points form the interior points, the later form are boundary points. In the picture below we see a surface which is a discrete cylinder. It has two boundary curves, both of which are circles. The orientation given on edges plays the role of orientations we use in the continuum when parametrizing curves. The function $\vec{F}$ on edges is a **vector field**. It describes how much is transported by the field from a node to each neighboring node. We have defined a gradient field $\nabla f$ of a scalar function $f$ as $\nabla f((a,b)) = f(b) - f(a)$ which is a function on edges, what is a vector field. We have exactly one week ago defined the curl of such a vector field and seen how to define the flux of of the curl through the surface. We have seen for example that the curl of a gradient field is zero. We also have defined line integrals in this frame work and learned how to compute it. By the way, the divergence of a vector field is a function on nodes again: it is the function $g(x) = \sum_{x \in e} F(e)$ counting how much goes in and out at a node. The operation which maps $f$ to $\Delta f = \text{divgrad}(f)$ plays the role of the Laplacian $\Delta f(x,y) = f_{xx} + f_{yy}$ in the continuum. It defines a finite $n \times n$ matrix if there are $n$ nodes. Its eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$ are the frequencies one can hear when using the network as a drum. Oliver worked the last 3 weeks during summer school on the problem whether $\sum_{j=1}^{k} \lambda_j \leq m + k(k+1)/2$, where $m$ is the number of edges. It is still an open problem called the **Brouwer conjecture**. Large enough regions as shown here satisfy the inequality.

Here is your problem: **Use Stokes theorem to compute the flux of the curl of $\vec{F}$ through $S$.** Also here, we do not just want to state the result, but see explanations and computation details.