"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by printing your name in the above box.

- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.

- Do not detach pages from this exam packet or unstaple the packet.

- Please try to write neatly. Answers which are illegible for the grader can not be given credit.

- No notes, books, calculators, computers, or other electronic aids are allowed.

- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.

- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are needed.

1) T F The graph $y = \sin(x)$ in the two dimensional plane is a curve parametrized by $\vec{r}(t) = [t, \sin(t)]$.

2) T F The vector fields $\text{curl}(\vec{F})$ and $\text{curl}(\vec{G})$ are perpendicular at every point, if $\vec{F}$ and $\vec{G}$ are perpendicular at every point in space.

3) T F The curvature of the curve $\vec{r}(t) = [3\sin(t), 0, 3\cos(t)]$ is 3 everywhere.

4) T F Any function that simultaneously satisfies the Laplace and wave equation must be linear.

5) T F Maxwell took a calculus exam, where he had to prove Stokes theorem.

6) T F The boundary of disk surface $\{x^4 + y^4 \le 1, z = 0\}$ is the curve $\{x^4 + y^4 = 1, z = 0\}$.

7) T F If $\vec{F}(x, y)$ is defined for all x, y and $\text{curl}(\vec{F}(x, y)) = 0$ everywhere, then $\vec{F}(x, y) = [f_x, f_y]$ is a gradient field.

8) T F In the movie "Rushmore", the "hardest problem of geometry" is solved using integration by parts.

9) T F We can use the fundamental theorem of line integrals to compute the area of a region.

10) T F The circles $[\cos(t), \sin(t), 0]$ and $[0, 1 + \cos(t), \sin(t)]$ are interlinked. You can not take them apart without them intersecting.

11) T F The quartic $-x^2 + y^2 - (z-1)^2 = 4$ is a two-sheeted hyperboloid.

12) T F If $\vec{r}(\phi, \theta)$ is the standard parametrization of a sphere, then the tangent vectors $\vec{r}_\phi$ and $\vec{r}_\theta$ are perpendicular.

13) T F The gradient of the divergence of $\vec{F} = [P, Q, R]$ defines a vector field that is always perpendicular to $\vec{F}$.

14) T F If $\vec{r}(t)$ parametrizes a curve, then the fourth derivative $\vec{r}\,''''(t)$ is called "snap".

15) T F For any bounded solid E in space, the boundary surface S has the property that it is a closed surface meaning that it does not have any boundary.

16) T F Given two vectors $\vec{v}, \vec{w}$, define $\vec{u} = \vec{v} \times \vec{w}$. It is always true that $\vec{w}, \vec{u}$ are perpendicular.

17) T F If $\vec{v} = \text{curl}(\vec{F})/|\text{curl}(\vec{F})|$ is defined and $g(x, y, z) = \text{div}(\vec{F})$, then the directional derivative is $D_{\vec{v}} g = 0$.

18) T F The vector projection of $\vec{F} = [P, Q, R]$ onto $[0, 0, 1]$ is $[0, 0, R]$.

19) T F For a given function f, define the new function $g = f_{xx} f_{yy} - f_{xy}^2$. Define $\vec{F} = \nabla g + \nabla f$, then the curl of $\vec{F}$ is zero.

20) T F The curvature of any curve on a sphere $x^2 + y^2 + z^2 = 1$ is always larger or equal to 1 at any point where the curvature is defined.
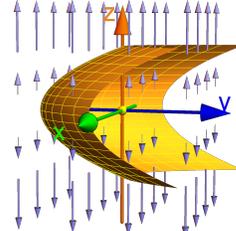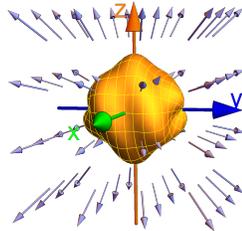
2

Problem 2) (10 points) No justifications are necessary.

a) (2 points) We decide in two cases whether the **flux** of the vector field through the surface S is $\boxed{\text{positive}}$ or $\boxed{\text{negative}}$. In the left picture (belonging to the boxes to the left), S is oriented outwards, in the right picture S is oriented in the positive y direction.

The flux through the left surface is

Positive

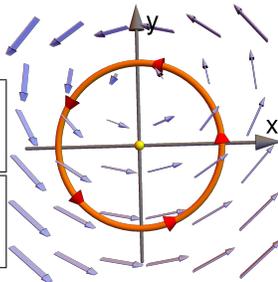| Negative |
| --- |

The flux through the right surface is

Positive

Negative

b) (2 points) Decide in each case, whether the **line integral** of the vector field along the closed circular loop is $\boxed{\text{positive}}$ or $\boxed{\text{negative}}$. Check the boxes on each side.

The line integral along the curve is
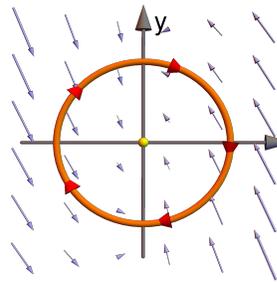
Positive

Negative

The line integral along the curve is
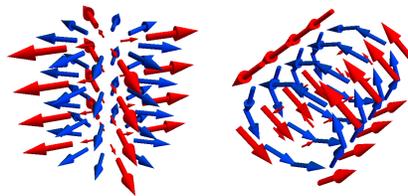
Positive

Negative

c) (2 points) Check on each side the box, for which the answer is positive $\boxed{> 0}$. Check exactly one box on the left for the left vector field and one box on the right for the right vector field.

For the field to the left: (check one box)

$|\text{curl}(\vec{F})| > 0$

$\text{div}(\vec{F}) > 0$

For the field to the right: (check one box)

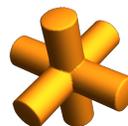| $|\text{curl}(\vec{F})| > 0$ |
| --- |

$\text{div}(\vec{F}) > 0$

d) (2 points) Remember that a solid E is **simply connected** if every closed loop in E be pulled together continuously to a point within E. Which are simply connected?

The solid to the left is (check one box)

simply connected

| not simply connected |

The solid to the right is (check one box)

simply connected

| not simply connected |

e) (2 points) Write down the names of two partial differential equations involving functions $f(t,x)$ for which only the first derivative with respect to t appear.

In a **2019 tictoc video** of Oliver, the **three fundamental derivative operations** div, curl and grad in multivariable calculus were investigated. There are 27 ways to combine three such operations. How many do make sense? In a) to c) you select the cases which **are defined**.

a) (3 points) In the following table, cross out every expression which is defined. Let $f(x, y, z)$ be a function of three variables.

grad(grad(grad(f)))	grad(curl(grad(f)))	grad(div(grad(f)))
curl(grad(grad(f)))	curl(curl(grad(f)))	curl(div(grad(f)))
div(grad(grad(f)))	div(curl(grad(f)))	div(div(grad(f)))

b) (3 points) In the following table, cross out every expression which **is defined**. Let $\vec{F} = [P, Q, R]$ denote a vector field in $\mathbf{R}^3$.

|		
grad(grad(curl($\vec{F}$)))	grad(curl(curl($\vec{F}$)))	grad(div(curl($\vec{F}$)))
curl(grad(curl($\vec{F}$)))	curl(curl(curl($\vec{F}$)))	curl(div(curl($\vec{F}$)))
div(grad(curl($\vec{F}$)))	div(curl(curl($\vec{F}$)))	div(div(curl($\vec{F}$)))

c) (3 points) In the following table, cross out every expression which **is defined**. Again, $\vec{F} = [P, Q, R]$ is a vector field in $\mathbf{R}^3$.

grad(grad(div($\vec{F}$)))	grad(curl(div($\vec{F}$)))	grad(div(div($\vec{F}$)))
curl(grad(div($\vec{F}$)))	curl(curl(div($\vec{F}$)))	curl(div(div($\vec{F}$)))
div(grad(div($\vec{F}$)))	div(curl(div($\vec{F}$)))	div(div(div($\vec{F}$)))

d) (1 point) Two of the following expressions are always zero (either the zero number or zero vector). Which ones? As before, $\vec{F} = [P, Q, R]$ is a vector field and $f(x, y, z)$ a scalar function in $\mathbf{R}^3$

curl(grad(f))	curl(curl($\vec{F}$))
div(grad(f))	div(curl(($\vec{F}$)

Oliver theorem from 2019 (see tictoc) tells that the number of ways in which one can combine n operations to make sense is given by the **Fibonacci number** $F(n + 3)$. In class, we had looked at the case $n = 2$, where $F(n + 3) = 5$ combinations made sense.
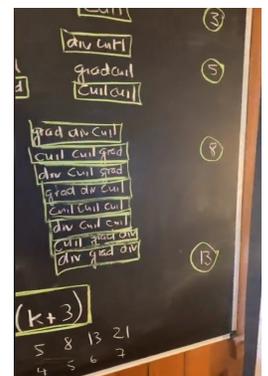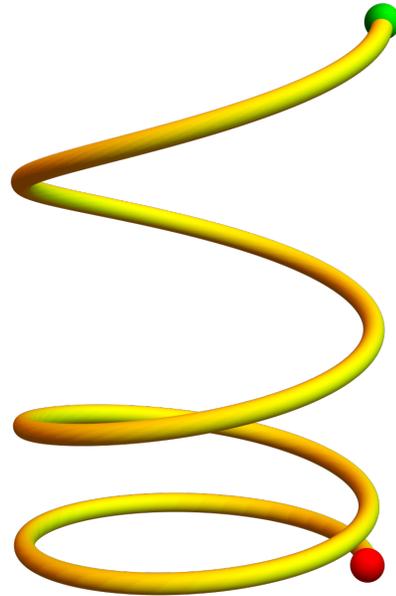

Screenshot from
Oliver's Tictoc 2019

Problem 4) (10 points)

A curve $\vec{r}(t)$ satisfies $\vec{r}\,'(t) = \begin{bmatrix} 2\cos(3t) \\ 2\sin(3t) \\ t \end{bmatrix}$ has the initial

point $\vec{r}(0) = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$.

a) (2 points) Locate the **end point** $\vec{r}(2\pi)$.

b) (2 points) Find the **acceleration** $\vec{r}\,''(0)$.

c) (2 points) Compute the **curvature** at $t = 0$.

d) (2 points) What is the **unit tangent vector** $\vec{T}(0)$?

e) (2 points) What is the **arc length** of the curve for $t \in [0, 2\pi]$?

Problem 5) (10 points)

a) (2 points) Find the **tangent plane** to the hyper-boloid $f(x, y, z) = x^2 + y^2 - z^2 = 1$ at $(1, 1, 1)$.

b) (2 points) Parametrize the **normal line** L to that plane passing through $(1, 1, 1)$.

c) (2 points) What is the **distance** of this normal line L to the origin $(0, 0, 0)$?

d) (2 points) **Estimate** the value $1.003^2 + 0.999^2 - 1.02^2$ using linear approximation (do not do it directly).

e) (2 points) What is the **directional derivative** $D_{\vec{v}} f$ at $(1, 1, 1)$ if $\vec{v} = \vec{i} = [1, 0, 0]$.
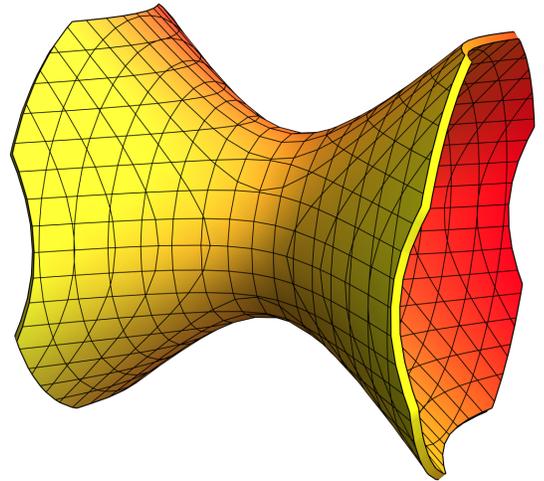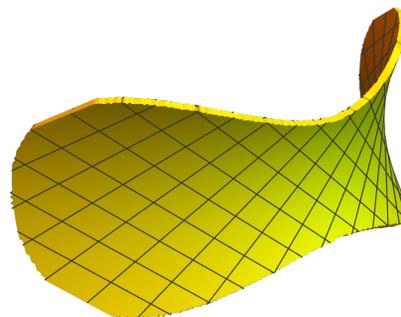
Problem 6) (10 points)

Find the surface area of the **travel sleep mask surface**

$$\vec{r}(u, v) = \begin{bmatrix} uv + 1 \\ u + v \\ u - v + 1 \end{bmatrix}$$

parametrized by the region $R = \{(u, v), u^2 + v^2 \leq 2\}$.

Problem 7) (10 points)

a) (5 points) Compute the double integral

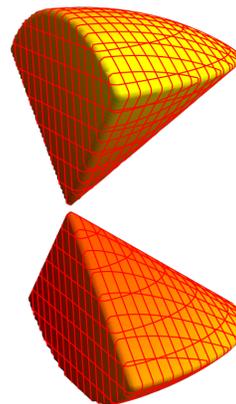$$\int_1^2 \int_0^{\sqrt{2-y}} \frac{\sin(\pi x)}{1 - x^2} \, dx dy \ .$$

b) (5 points) Evaluate the integral

$$\iiint_E (x^2 + y^2 + z^2)^4 \, dV \ ,$$

where

$$E = \{(x, y, z), x^2 + y^2 + z^2 \le 1, z^2 \ge x^2 + y^2, y \ge 0\} \ .$$

The solid E is displayed to the right.

Problem 8) (10 points)

Since we noticed some **log identity deficiencies** in the midterm, lets see whether we have made our homework and learned from our **mistakes**. Logs again!

Find the maxima and minima of

$$f(x,y) = \log(x) + 2\log(y) - x - y$$

on $x > 0, y > 0$ and classify them using the second derivative test. As usual $\log(x) = \ln(x)$ is the natural log.
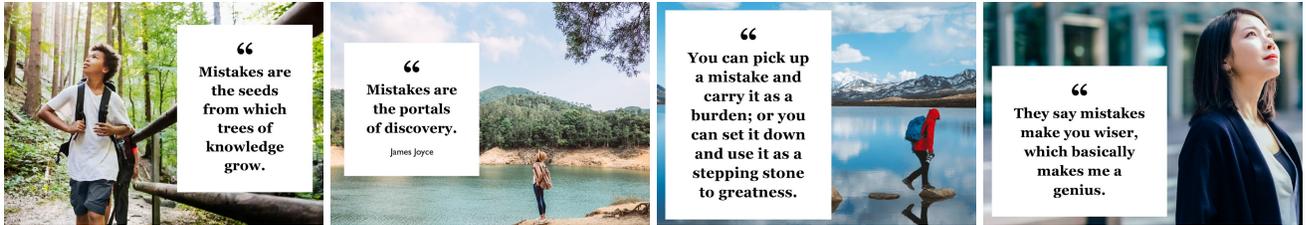

Image credit: **Kristin McKarty**, "35 quotes about learning from your mistakes to reassure you", Nov. 17, 2021, accessed from Lovetoknow.com, Aug. 6. 2024

Problem 9) (10 points)

A Lagrange problem can be looked at from two different perspectives: either extremize f under the constraint $g = c$ or extremize g under the constraint $f = d$. It is a **Janus face**, a symbol of **duality**.

a) (5 points) Use the Lagrange method to find the maximum of
$$f(x, y) = xy$$
under the constraint
$$g(x, y) = x + 3y = 6 \ .$$

b) (5 points) Now use the Lagrange method to find the minimum of
$$f(x, y) = x + 3y$$
under the constraint $xy = 3$.

Statue of Janus, the god of all beginnings, gates, transitions, time, choices, duality, doorways, passages and endings. Image source: Wikipedia
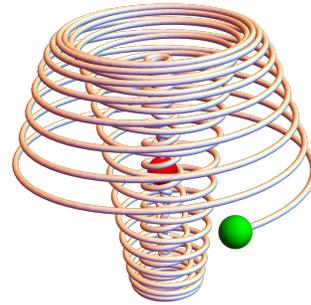
Problem 10) (10 points)

This year features the movie **"twisters"**, which has been quite well received and which Oliver still wants to see, once summer school is over. Compute the line integral

$$\int_C \vec{F} \cdot \vec{dr}$$

along the **twister curve** C parametrized as

$$\vec{r}(t) = \begin{bmatrix} e^{-t}\cos(100t) \\ e^{-t}\sin(100t) \\ e^{-t/100}\sin(t) \end{bmatrix}$$

from $t = 0$ to $t = \infty$. It connects $A = \vec{r}(0)$ with $B = \vec{r}(\infty) = (0,0,0)$. The vector field is

$$\vec{F}(x,y,z) = \begin{bmatrix} 10x^4 \\ 6y^2 + z \\ 3z^2 + y \end{bmatrix}.$$

Problem 11) (10 points)

We place the **Stanford bunny** E into a radiation field $\vec{F}$. The bunny is the most famous object in computer science. Its skin S is a closed surface. We assume here that S is oriented **inwards**. You enter the Mathematica command line

E=DiscretizeGraphics[ExampleData[{"Geometry3D", "StanfordBunny"}]];

NIntegrate[1,{x,y,z} ∈ E] which gives you the output 0.057. What is the flux

$$\iint_S \vec{F} \cdot d\vec{S}$$

of the field

$$\vec{F}(x, y, z) = \begin{bmatrix} 500x + y + z \\ 100x + 100z \\ x + y + 500z \end{bmatrix}$$
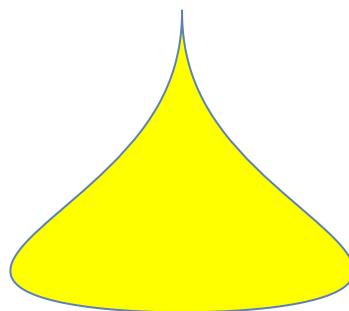
through the **Stanford Bunny surface** S?

Problem 12) (10 points)

Find the area of the **honey-drop region** enclosed by

$$\vec{r}(t) = \begin{bmatrix} \frac{\sin^2(\pi t)}{t} \\ 4t^2 - 4 \end{bmatrix}$$

parametrized on the interval $-1 \le t \le 1$.

Find the flux

$$\iint_S \mathrm{curl}(\vec{F}) \cdot d\vec{S}$$

of the curl of the vector field

$$\vec{F}(x, y, z) = \begin{bmatrix} -yz^2 \\ xz^2 \\ x^4 y^6 z^9 \sin(e^{xyz}) \end{bmatrix}$$

through the outwards oriented surface

$$S = \{(x, y, z), \,|\, x^2 + y^2 + z^2 = 4 \text{ and } z \leq \sqrt{3}\} .$$