8/7/2025 PRACTICE 4 **Maths 21a, O.Knill, Summer 2025**

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

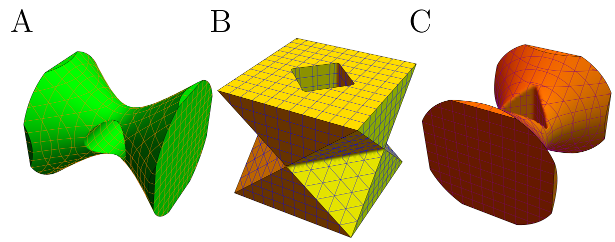
Problem 1) (20 points) No justifications are needed.

1)	T	F	The surface parametrized by $\vec{r}(u,v) = [u, v, u^2 - v^2]$ is a hyperbolic paraboloid.
2)	T	F	The vector $[3,5,5]$ is perpendicular to the vector $[0,2,-2]$.
3)	T	F	The surface $(x-1)^2 - y^2 + z^2 - 2z = 1$ is a one sheeted hyperboloid.
4)	T	F	The gradient vector $\text{grad}(f)(x,y,z) = \nabla f(x,y,z)$ is parallel to $\text{grad}(\text{div}(\text{grad}(f)))(x,y,z)$.
5)	T	F	If $\lvert\vec{u} \times \vec{v}\rvert = \vec{u} \cdot \vec{v}$ holds, then the angle α between $\vec{u}$ and $\vec{v}$ satisfies $\alpha = \pi/4$.
6)	T	F	The line integral of $\vec{F}(x,y,z) = [x,y,z]$ along the curve $\vec{r}(t) = [\cos(t), \sin(2t), \sin(t)]$ for $0 \leq t \leq 2\pi$ is zero.
7)	T	F	If $f(t,x)$ solves the differential equation equation $u_{ttt} = u_{xxx}$ then $f(2t, 2x)$ solves the same differential equation.
8)	T	F	The curvature of the curve $[\cos(7t), 0, \sin(7t)]$ at $t = 0$ is the same as the curvature of the curve $[\cos(3t), 0, \sin(3t)]$ at $t = 0$.
9)	T	F	The equation $x + y = 3$ in three dimensional space is a plane.
10)	T	F	The vector projection of $[3,1,1]$ onto $[0,0,7]$ is $[0,0,1]$.
11)	T	F	If $\vec{r}(t)$ is the circle $[\cos(t), \sin(t), 1]$, the binormal vector $\vec{B}$ to this curve satisfies $\vec{B}(t) = [0,0,1]$ for all t.
12)	T	F	If $f(x,y,z)$ is a function and $\vec{F} = \nabla f$ then $\text{div}(\vec{F}) = 0$ everywhere.
13)	T	F	For any function vector field $\vec{F}$, we have $\text{curl}(\text{curl}(\vec{F})) = \vec{0}$.
14)	T	F	For all vector fields $\vec{F}$ we have $\text{curl}(\vec{F}) = \text{grad}(\text{div}(\vec{F}))$.
15)	T	F	If $\vec{F}$ is a gradient field and S is a closed surface then $\iint_S \vec{F} \cdot d\vec{S} = 0$.
16)	T	F	If $\vec{u}, \vec{v}, \vec{w}$ are unit space vectors, then the length of the projection of $\vec{u} \times \vec{v}$ onto $\vec{w}$ is the same as the length of the projection of $\vec{v} \times \vec{w}$ onto $\vec{u}$.
17)	T	F	There exists a vector field $\vec{F}(x,y,z)$ in space such that $\text{curl}(\vec{F}) = [x, -3y, z]$.
18)	T	F	The distance between the line $\vec{r}(t) = [t,t,t]$ and the point $(3,4,5)$ is $\lvert[2,3,4] \times [1,1,1]\rvert/\sqrt{3}$.
19)	T	F	The sphere $x^2 + y^2 + z^2 = 1$ without north and south pole is a simply connected surface.
20)	T	F	If the directional derivatives of $f(x,y,z)$ at $(0,0,0)$ in the directions $\vec{u} = [1,1,1]/\sqrt{3}, \vec{v} = [1,1,0]/\sqrt{2}, \vec{w} = [1,0,0]$ are known, then the gradient of f at $(0,0,0)$ is determined.

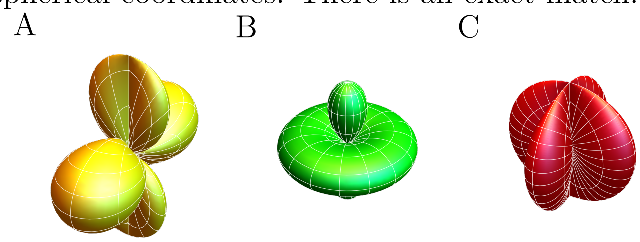
Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the solids. There is an exact match.

Solid	A-C
$x^2 - y^2 + z^2 < 2, \lvert x\rvert + \lvert y\rvert > 1.3$	
$\lvert x\rvert + \lvert y\rvert - \lvert z\rvert < 2, \lvert x\rvert + \lvert y\rvert > 1$	
$-x^2 + y^2 + z^2 < 1.5, x^2 + z^2 > 1$	



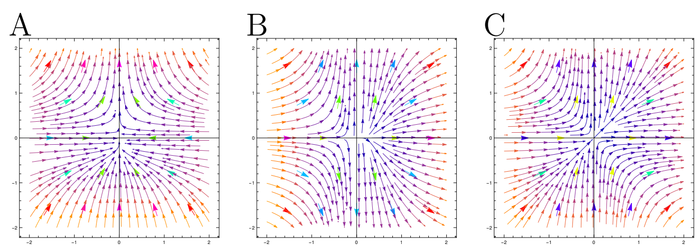
b) (2 points) Match the surfaces given in spherical coordinates. There is an exact match.



Surface	A-C
$\rho = \cos(2\theta)$	
$\rho = \cos(2\phi)$	
$\rho = \cos(2\theta) + \cos(2\phi)$	

c) (2 points) The figures display vector fields in the plane. There is an exact match.

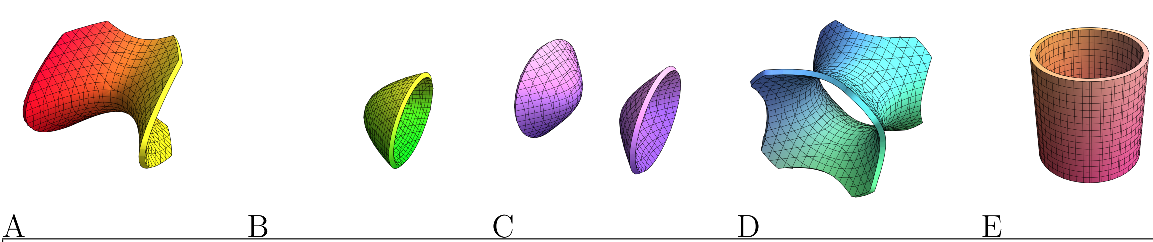
Field	A-C
$\vec{F} = [x^2, y]$	
$\vec{F} = [-x, y^2]$	
$\vec{F} = [x^2, y^2]$	



d) (2 points) State the name of a partial differential equation different from the heat equation, wave equation, transport equation, Laplace equation or Black-Scholes equation. (Don't use a joke like the "Knill equation". The differential equation should be generally known).

e) (2 points) Just to double check that you know your quadrics

	Enter a letter from A-E in each of the three cases
Pick the two sheeted hyperboloid	
Pick the elliptic paraboloid	
Pick the cylinder	



Problem 3) (10 points) No justifications necessary

The following questions, like the T/F problems probe some deeper understanding of the theory. Most of the questions have been discussed in class. In each of the questions, check exactly one of the three answers.

a) (2 points) What can you say about the second derivative test in the case $D > 0$ and $f_{xx} = 0$.

We have a saddle	The situation does not occur	It is either a maximum or minimum

b) (2 points) What happens in the Lagrange equations for a function f with constraint $g = c$ if in the solution the Lagrange multiplier is $\lambda = 0$?

We have an inflection point	The situation does not occur	We have a critical point of f.

c) (2 points) If $\vec{F} = [y, -x]$ is a vector field and C is a curve C bounding $x^4 + y^4 \leq 1$ oriented counter clockwise. What can you say in general about the sign of $\int_C \vec{F} \cdot \vec{dr}$?

It is positive	It is negative	It is zero

d) (2 points) If $g = x^6 + y^6 + z^6$ and $\vec{F} = \nabla g$ for a scalar function g. What can you say in general about the sign of the flux of the vector field $\vec{F}$ through the contour surface $g = 1$, if S is oriented outwards?

It is positive	It is negative	It is zero

e) (2 points) Given the vector field $\vec{G} = \text{curl}(\vec{F})$. What can you say in general about the flux of $\vec{G}$ through the torus surface S, if it is oriented outwards?

It is positive	It is negative	It is zero

Problem 4) (10 points)

We are describing a curve $\vec{r}(t)$ satisfying $\vec{r}\,''(t) = [0, 0, 2]$ for all t, and satisfying the condition $\vec{r}\,'(0) = [2, 0, 0]$ and $\vec{r}\,(0) = [2, 0, 0]$.

a) (2 points) What is the unit tangent vector $\vec{T}(0)$ at $t = 0$?

b) (2 points) Compute $\int_0^1 \vec{r}\,'(t)\;dt$.

c) (2 points) Compute $\int_0^1 |\vec{r}\,'(t)|^2\;dt$.

d) (2 points) Compute $\int_0^1 |\vec{r}\,''(t)|\;dt$.

e) (2 points) What is the curvature of the curve at $\vec{r}(1)$?

Problem 5) (10 points)

a) (3 points) Find the tangent line to $f(x,y) = x^2y^2 - x^3 + y^3 = 1$ at $(x,y) = (1,1)$.

b) (3 points) Estimate $f(1.1, 0.8)$ using linearization.

c) (4 points) Find the directional derivative $D_{[3,4]/5}f$ at $(1,1)$.

Problem 6) (10 points)

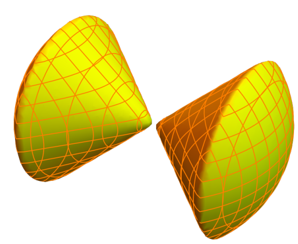
We are given the three points $A = (1,2,3), B = (2,3,4), C = (5,4,3)$ in space.

a) (4 points) Find the **surface area** of the triangle ABC using the surface area formula, that means by parametrizing the triangle $\vec{r}(s,t)$ as part of a plane and using the surface area formula. As always, we want to see this formula.

b) (3 points) Compute the surface area of the triangle using one of the products you know. As always, we want to see the formula you use.

c) (3 points) Find the distance of the point C to the line that passes through A, B. Also here, as always, we want to see the general formula of a point to a line.

Problem 7) (10 points)

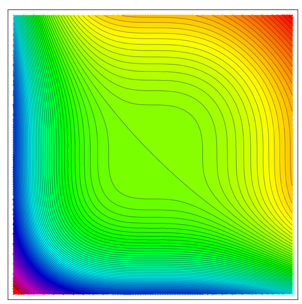


Integrate the function

$$f(x,y,z) = (x^2 + y^2 + z^2)^4$$

over the part of the solid cone $y^2 + z^2 \leq x^2$ satisfying $x^2 + y^2 + z^2 \leq 4$.

Problem 8) (10 points)



The picture shows contour lines of g(x,y)

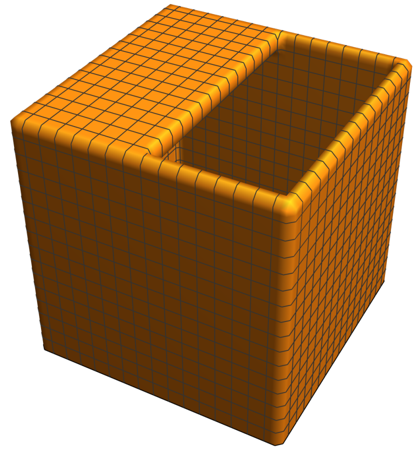
Let $g(x,y) = x^4 - 4x^3 + y^4 - 4y^3$. First verify that the function $f(x,y) = \Delta g(x,y) = g_{xx}(x,y) + g_{yy}(x,y)$ is the function

$$f(x,y) = -24x + 12x^2 - 24y + 12y^2 \; .$$

Now classify the critical points of this function $f(x,y)$ using the second derivative test.

Problem 9) (10 points)

A **pepper box** of height y and base length $2x$ and base width $2x$. It is half open on the top so that the (weighted) surface area is $f(x,y) = 6x^2 + 4xy$. Find the parameters x, y which maximize the area, provided that the total rim length $18x + 4y = 48$ is fixed. Of course, you must show us that you master the Lagrange method.



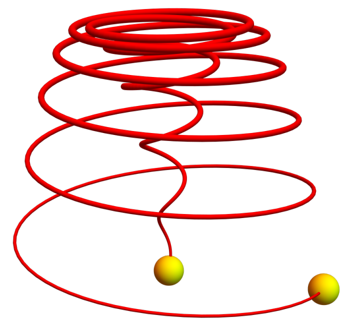
Problem 10) (10 points)

Compute the line integral $\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$ for the curve

$$\vec{r}(t) = [t\cos(10t), t\sin(10t), t(2\pi - t)]$$

with $0 \leq t \leq 2\pi$ and the vector field

$$\vec{F}(x,y,z) = [y + 6x^5, x + 8y^7, \sin(z)]\ .$$

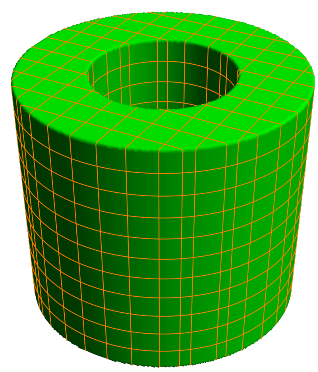


Problem 11) (10 points)

Find the flux of the vector field

$$\vec{F}(x,y,z) = [x^4, -4yx^3, 7z + \sin(xe^y)]$$

through the outwards oriented surface which is the boundary of the solid $1 \leq x^2 + y^2 < 4$, $1 \leq z \leq 3$.

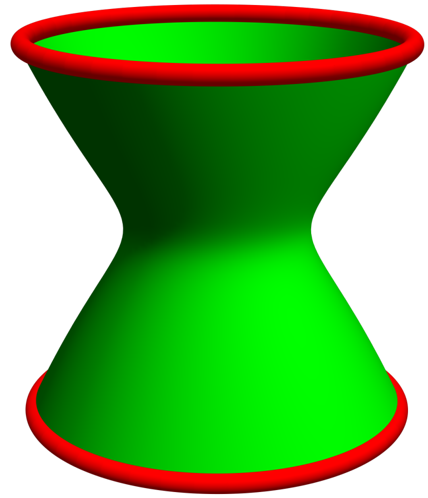


Problem 12) (10 points)

Find the flux of the curl of the vector field

$\vec{F}(x,y,z) = [(z^2-4)\cos(xyz), 3x+y, (z^2-4)\sin(e^x+y^2+z)]$

through the surface $(x^2+y^2)^2 = 4+3z^2$, $-2 \le z \le 2$. The surface is oriented outwards.



Problem 13) (10 points)

Find the area of the region bound by the closed curve

$$\vec{r}(t) = [\cos^3(t) + \cos(t), \sin(t)]$$

with $0 \le t \le 2\pi$.