8/7/2025 PRACTICE 4 **Maths 21a, O.Knill, Summer 2025**

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are needed.

1) T F The surface parametrized by $\vec{r}(u,v) = [u, v, u^2 - v^2]$ is a hyperbolic paraboloid.

Solution:
It is a graph $z = x^2 - y^2$.

2) T F The vector $[3, 5, 5]$ is perpendicular to the vector $[0, 2, -2]$.

Solution:
Compute the dot product.

3) T F The surface $(x-1)^2 - y^2 + z^2 - 2z = 1$ is a one sheeted hyperboloid.

Solution:
Complete the square

4) T F The gradient vector $\mathrm{grad}(f)(x,y,z) = \nabla f(x,y,z)$ is parallel to $\mathrm{grad}(\mathrm{div}(\mathrm{grad}(f)))(x,y,z)$.

Solution:
Take $f(x,y,z) = x^3 + y$ for example.

5) T F If $|\vec{u} \times \vec{v}| = \vec{u} \cdot \vec{v}$ holds, then the angle α between $\vec{u}$ and $\vec{v}$ satisfies $\alpha = \pi/4$.

Solution:
When we talk about angle, we assume it exists. And then indeed, this means $\sin(\alpha) = \cos(\alpha)$ which forces $\alpha = \pi/4$.

6) T F The line integral of $\vec{F}(x,y,z) = [x, y, z]$ along the curve $\vec{r}(t) = [\cos(t), \sin(2t), \sin(t)]$ for $0 \leq t \leq 2\pi$ is zero.

Solution:
It is a gradient field and the curve is closed.

7) T F

If $f(t,x)$ solves the differential equation equation $u_{ttt} = u_{xxx}$ then $f(2t,2x)$ solves the same differential equation.

Solution:
Use the chain rule. Indeed in both cases, we get a factor 8.

8) T F

The curvature of the curve $[\cos(7t), 0, \sin(7t)]$ at $t = 0$ is the same as the curvature of the curve $[\cos(3t), 0, \sin(3t)]$ at $t = 0$.

Solution:
Both curves are a circle of radius 1.

9) T F

The equation $x + y = 3$ in three dimensional space is a plane.

Solution:
Indeed, it is a n equation of the form $ax + by + cz = d$.

10) T F

The vector projection of $[3, 1, 1]$ onto $[0, 0, 7]$ is $[0, 0, 1]$.

Solution:
Just compute it. It is $[3, 1, 1] \cdot [0, 0, 7]/7^2[0, 0, 7] = [0, 0, 1]$.

11) T F

If $\vec{r}(t)$ is the circle $[\cos(t), \sin(t), 1]$, the binormal vector $\vec{B}$ to this curve satisfies $\vec{B}(t) = [0, 0, 1]$ for all t.

Solution:
Indeed, it has to be perpendicular to the plane.

12) T F

If $f(x, y, z)$ is a function and $\vec{F} = \nabla f$ then $\text{div}(\vec{F}) = 0$ everywhere.

Solution:
For example $f(x, y, z) = x^2/2$, then $\vec{F} = [x, 0, 0]$ and $\text{div}\vec{F} = 1$.

13) T **F** For any function vector field $\vec{F}$, we have $\text{curl}(\text{curl}(\vec{F})) = \vec{0}$.

Solution:
Not necessarily.

14) T **F** For all vector fields $\vec{F}$ we have $\text{curl}(\vec{F}) = \text{grad}(\text{div}(\vec{F}))$.

Solution:
Already a simple example $[-y, x, 0]$ shows that it is false.

15) T **F** If $\vec{F}$ is a gradient field and S is a closed surface then $\iint_S \vec{F} \cdot d\vec{S} = 0$.

Solution:
This follows from the divergence theorem or from Stokes theorem.

16) **T** F If $\vec{u}, \vec{v}, \vec{w}$ are unit space vectors, then the length of the projection of $\vec{u} \times \vec{v}$ onto $\vec{w}$ is the same as the length of the projection of $\vec{v} \times \vec{w}$ onto $\vec{u}$.

Solution:
In both cases, we have the same triple scalar product.

17) T **F** There exists a vector field $\vec{F}(x, y, z)$ in space such that $\text{curl}(\vec{F}) = [x, -3y, z]$.

Solution:
The divergence would have to be zero.

18) **T** F The distance between the line $\vec{r}(t) = [t, t, t]$ and the point $(3, 4, 5)$ is $|[2, 3, 4] \times [1, 1, 1]|/\sqrt{3}$.

Solution:
It is the formula.

19) T F The sphere $x^2 + y^2 + z^2 = 1$ without north and south pole is a simply connected surface.

Solution:
You can not deform pull the equator together to a point.

20) T F If the directional derivatives of $f(x, y, z)$ at $(0, 0, 0)$ in the directions $\vec{u} = [1, 1, 1]/\sqrt{3}, \vec{v} = [1, 1, 0]/\sqrt{2}, \vec{w} = [1, 0, 0]$ are known, then the gradient of f at $(0, 0, 0)$ is determined.

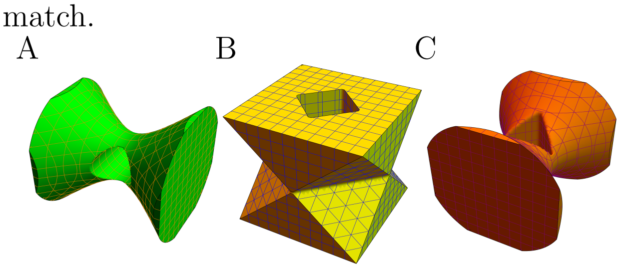
Solution:
One can determine the gradient $[a, b, c]$.

Solution:
False are 4,12-15,17,19. All other are true.

Problem 2) (10 points) No justifications are necessary.

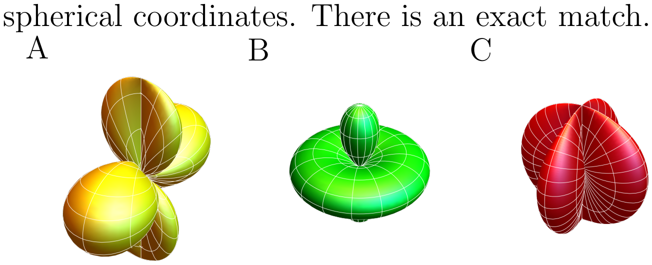
a) (2 points) Match the solids. There is an exact match.

Solid	A-C
$x^2 - y^2 + z^2 < 2, \|x\| + \|y\| > 1.3$	
$\|x\| + \|y\| - \|z\| < 2, \|x\| + \|y\| > 1$	
$-x^2 + y^2 + z^2 < 1.5, x^2 + z^2 > 1$	



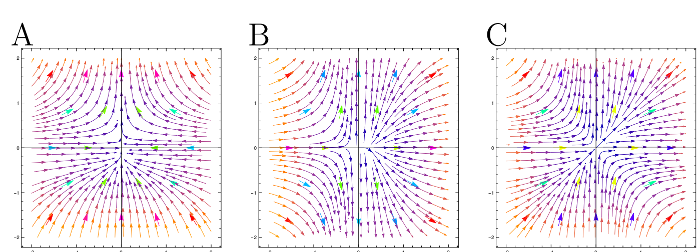
b) (2 points) Match the surfaces given in spherical coordinates. There is an exact match.

Surface	A-C
$\rho = \cos(2\theta)$	
$\rho = \cos(2\phi)$	
$\rho = \cos(2\theta) + \cos(2\phi)$	



c) (2 points) The figures display vector fields in the plane. There is an exact match.

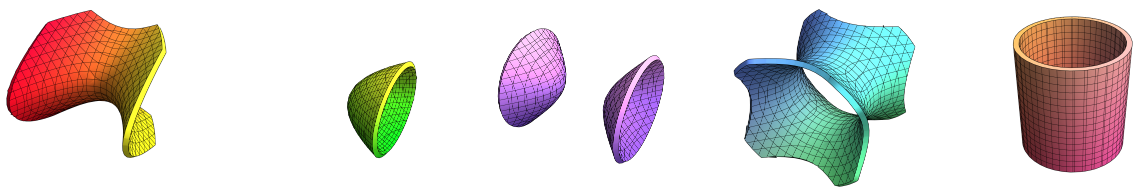
Field	A-C
$\vec{F} = [x^2, y]$	
$\vec{F} = [-x, y^2]$	
$\vec{F} = [x^2, y^2]$	



d) (2 points) State the name of a partial differential equation different from the heat equation, wave equation, transport equation, Laplace equation or Black-Scholes equation. (Don't use a joke like the "Knill equation". The differential equation should be generally known).

e) (2 points) Just to double check that you know your quadrics

	Enter a letter from A-E in each of the three cases
Pick the two sheeted hyperboloid	
Pick the elliptic paraboloid	
Pick the cylinder	



A B C D E

Solution:
a) CBA
b) CBA
c) BAC
d) Burgers or Eiconal or Schroedinger or Clairaut are examples.
e) CBE

Problem 3) (10 points) No justifications necessary

The following questions, like the T/F problems probe some deeper understanding of the theory. Most of the questions have been discussed in class. In each of the questions, check exactly one of the three answers.

a) (2 points) What can you say about the second derivative test in the case $D > 0$ and $f_{xx} = 0$.

We have a saddle	The situation does not occur	It is either a maximum or minimum

b) (2 points) What happens in the Lagrange equations for a function f with constraint $g = c$ if in the solution the Lagrange multiplier is $\lambda = 0$?

We have an inflection point	The situation does not occur	We have a critical point of f.

c) (2 points) If $\vec{F} = [y, -x]$ is a vector field and C is a curve C bounding $x^4 + y^4 \leq 1$ oriented counter clockwise. What can you say in general about the sign of $\int_C \vec{F} \cdot \vec{dr}$?

It is positive	It is negative	It is zero

d) (2 points) If $g = x^6 + y^6 + z^6$ and $\vec{F} = \nabla g$ for a scalar function g. What can you say in general about the sign of the flux of the vector field $\vec{F}$ through the contour surface $g = 1$, if S is oriented outwards?

It is positive	It is negative	It is zero

e) (2 points) Given the vector field $\vec{G} = \text{curl}(\vec{F})$. What can you say in general about the flux of $\vec{G}$ through the torus surface S, if it is oriented outwards?

It is positive	It is negative	It is zero

Solution:
a) Not defined.
b) Critical points.
c) Negative
d) Positive
e) Zero

Problem 4) (10 points)

We are describing a curve $\vec{r}(t)$ satisfying $\vec{r}\,''(t) = [0, 0, 2]$ for all t, and satisfying the condition $\vec{r}\,'(0) = [2, 0, 0]$ and $\vec{r}\,(0) = [2, 0, 0]$.

a) (2 points) What is the unit tangent vector $\vec{T}(0)$ at $t = 0$?

b) (2 points) Compute $\int_0^1 \vec{r}\,'(t)\; dt$.

c) (2 points) Compute $\int_0^1 |\vec{r}\,'(t)|^2\; dt$.

d) (2 points) Compute $\int_0^1 |\vec{r}\,''(t)|\; dt$.

e) (2 points) What is the curvature of the curve at $\vec{r}(1)$?

Solution:
It is helpful to first integrate twice to get $\vec{r}(t) = [2t+2, 0, t^2]$ and $\vec{r}\,'(t) = [2, 0, 2t]$.
a) $[1, 0, 0]$
b) $[2, 0, 1]$
c) $16/3$
d) 2
e) $1/(4\sqrt{2})$

Problem 5) (10 points)

a) (3 points) Find the tangent line to $f(x, y) = x^2y^2 - x^3 + y^3 = 1$ at $(x, y) = (1, 1)$.

b) (3 points) Estimate $f(1.1, 0.8)$ using linearization.

c) (4 points) Find the directional derivative $D_{[3,4]/5}f$ at $(1, 1)$.

Solution:
a) The gradient is $[a, b] = [-1, 5]$. The equation of the line is $ax + by = d$ and so $\boxed{-x + 5y = 4}$, where the constant d had been obtained by plugging in $(1, 1)$.
b) $L(1.1, 0.8) = 1 + (-1)(1.1 - 1) + 5(0.8 - 1) = \boxed{-0.1}$.
c) $[a, b] \cdot [3, 4]/5 = [-1, 5] \cdot [3, 4]/5 = (-3 + 20)/5 = \boxed{17/5}$.

Problem 6) (10 points)

We are given the three points $A = (1, 2, 3), B = (2, 3, 4), C = (5, 4, 3)$ in space.

a) (4 points) Find the **surface area** of the triangle ABC using the surface area formula, that means by parametrizing the triangle $\vec{r}(s, t)$ as part of a plane and using the surface area formula. As always, we want to see this formula.

b) (3 points) Compute the surface area of the triangle using one of the products you know. As always, we want to see the formula you use.

c) (3 points) Find the distance of the point C to the line that passes through A, B. Also here, as always, we want to see the general formula of a point to a line.

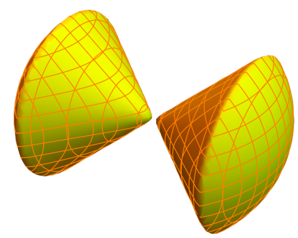
Solution:
a) Parametrize $\vec{r}(t, s) = [1+t, 4s, 2+t+2s, 3+t]$ compute $|\vec{r}_t \times \vec{r}_s|$ and integrate $\int_0^1 \int_0^t |\vec{r}_t \times \vec{r}_s| \; dsdt = \sqrt{6}$.
b) $|\vec{AB} \times \vec{AC}|/2 = \sqrt{6}$.
c) $|\vec{AB} \times \vec{AC}|/|\vec{AB}| = 2\sqrt{2}$.

Problem 7) (10 points)

Integrate the function

$$f(x, y, z) = (x^2 + y^2 + z^2)^4$$

over the part of the solid cone $y^2 + z^2 \leq x^2$ satisfying $x^2 + y^2 + z^2 \leq 4$.



Solution:
We need spherical coordinates (treating the x-axes as what we usually take as z-axes) and take twice the volume of one.

$$2 \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho^8 \rho^2 \sin(\phi) \; d\rho d\phi d\theta$$

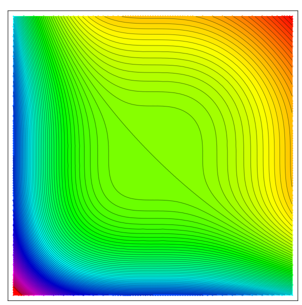
which is $\boxed{4\pi \frac{2^{11}}{11}(\frac{\sqrt{2}}{2} - 1)}$.

Problem 8) (10 points)

Let $g(x, y) = x^4 - 4x^3 + y^4 - 4y^3$. First verify that the function $f(x, y) = \Delta g(x, y) = g_{xx}(x, y) + g_{yy}(x, y)$ is the function

$$f(x, y) = -24x + 12x^2 - 24y + 12y^2 \; .$$

Now classify the critical points of this function $f(x, y)$ using the second derivative test.

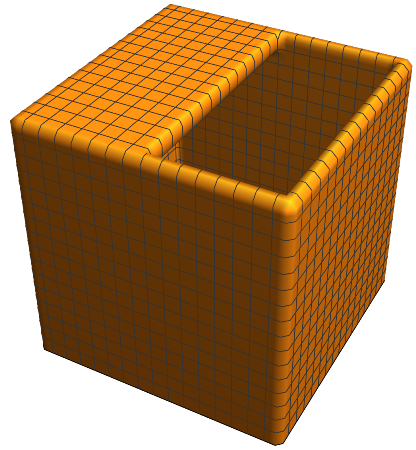


The picture shows contour lines of g(x,y)

Solution:
The computation $g_{xx} + g_{yy} = f(x, y)$ requires taking partial derivative. The gradient is $\nabla f = [24x - 24, 24y - 24]$ which is zero for $\boxed{(x, y) = (1, 1)}$. As $D = 24^2$ and $f_{xx} = 24$ we have a $\boxed{\text{minimum}}$.

Problem 9) (10 points)

A **pepper box** of height y and base length $2x$ and base width $2x$. It is half open on the top so that the (weighted) surface area is $f(x,y) = 6x^2 + 4xy$. Find the parameters x, y which maximize the area, provided that the total rim length $18x + 4y = 48$ is fixed. Of course, you must show us that you master the Lagrange method.



Solution:
The Lagrange equations are $12x + 4y = \lambda 18, 4x = \lambda 4, 18x + 4y = 48$. Solving them gives $\boxed{(x,y) = (2,3)}$.

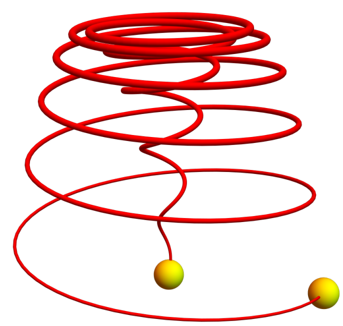
Problem 10) (10 points)

Compute the line integral $\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$ for the curve

$$\vec{r}(t) = [t\cos(10t), t\sin(10t), t(2\pi - t)]$$

with $0 \leq t \leq 2\pi$ and the vector field

$$\vec{F}(x,y,z) = [y + 6x^5, x + 8y^7, \sin(z)]\ .$$



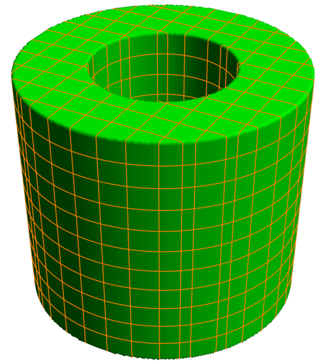
Solution:
Use the fundamental theorem of line integrals. The vector field is a gradient field $\vec{F} = \nabla f$ with $f(x,y,z) = xy + x^6 + y^8 - \cos(z)$. We have $\vec{r}(2\pi) = [2\pi, 0, 0]$ and $\vec{r}(0) = [0,0,0]$. The FTLI tells $\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$ which is $f(2\pi,0,0) - f(0,0,0) = \boxed{(2\pi)^6}$.

Problem 11) (10 points)

Find the flux of the vector field

$$\vec{F}(x,y,z) = [x^4, -4yx^3, 7z + \sin(xe^y)]$$

through the outwards oriented surface which is the boundary of the solid $1 \leq x^2 + y^2 < 4$, $1 \leq z \leq 3$.



Solution:
Use the divergence theorem. We have $\text{div}(\vec{F}) = 7$. The divergence theorem is $\iiint_E 7\, dV = 7\text{Vol}(E) = 7(4\pi - \pi)2 = \boxed{42\pi}$.

Problem 12) (10 points)

Find the flux of the curl of the vector field

$$\vec{F}(x,y,z) = [(z^2-4)\cos(xyz), 3x+y, (z^2-4)\sin(e^x + y^2 + z)]$$

through the surface $(x^2+y^2)^2 = 4 + 3z^2$, $-2 \leq z \leq 2$. The surface is oriented outwards.

Solution:
We use Stokes theorem. The surface has 2 boundary parts. One is parametrized as $\vec{r}_1(t) = [2\cos(t), 2\sin(t), 2]$, the other (is oriented in the opposite direction!) $\vec{r}_2(t) = [2\cos(t), -2\sin(t), -2]$. The line integrals are

$$\int_0^{2\pi} [0, 6\cos(t) + 2\sin(t), 0] \cdot [-2\sin(t), 2\cos(t), 0]\, dt$$

$$+ \int_0^{2\pi} [0, 6\cos(t) - 2\sin(t), 0] \cdot [-2\sin(t), -2\cos(t), 0]\, dt$$

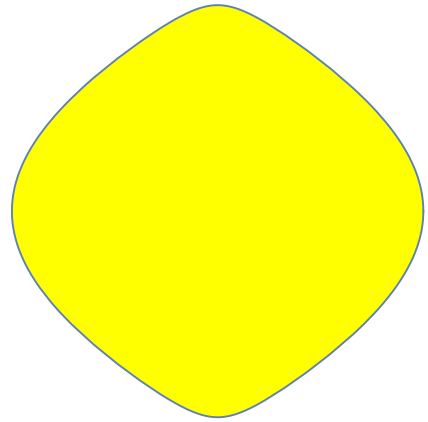
which is $12\pi - 12\pi = 0$. The answer is $\boxed{0}$.

Problem 13) (10 points)

Find the area of the region bound by the closed curve

$$\vec{r}(t) = [\cos^3(t) + \cos(t), \sin(t)]$$

with $0 \le t \le 2\pi$.



Solution:
Use Green's theorem with $\vec{F} = [0, x]$ so that the area of the region is

$$\iint_G 1\ dA = \iint_G \text{curl}(F)\ dA = \int_C \vec{F}\ d\vec{r}$$

The line integral is

$$\int_0^{2\pi} [0, \cos^3(t) + \cos(t)], [-3\cos^2(t)\sin(t) - \sin(t), \cos(t)]\ dt = \int_0^{2\pi} \cos^4(t) + \cos^2(t)\ dt$$

This integral can be solved using the double angle formula $\cos^2(t) = (1 + \cos(2t))/2$ so that $\cos^4(t) = (1 + \cos(2t))^2/4 = [1 + 2\cos(2t) + (1 + \cos(4t)/2]/4$. The integral is $2\pi(1/2 + 1/4 + 1/8) = \boxed{7\pi/4}$.