"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are needed.

1) T **F** The vector $[0, 2, 1]/5$ is a direction vector.

Solution:
We would have to divide by the square root. Was overlooked by many!

2) **T** F Let P be a point and U, V be two spheres. The distances satisfy the inequality $d(P, U) + d(P, V) \geq d(U, V)$.

Solution:
This is a harder problem. If we take the analogue for spheres, then it is wrong. It is however true if the third sphere is a point. The reason is that the shortest distance between two spheres can be obtained from the distance of the centers and the radii (depending on the constellation). In any case, the shortest distance is always the length of a line segment connecting one sphere with the other and that this line segment has to go through the origins of the spheres. Now, if P is on that line segment, then the equality $d(P, U) + d(P, V) = d(U, V)$ holds. In general, the sum of the distances is by the triangle inequality larger. I will make an exhibit about this.

3) T **F** At a local minimum (x_0, y_0) of $f(x, y)$ we always have $f_{xx}(x_0, y_0) > 0$.

Solution:
It can be a case with $D = 0$. An example is $x^4 + y^4$. Also this was marked wrong by most.

4) **T** F For the unit sphere S and $\vec{F} = [x^2yz, xy, xz]$, the flux $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$ is zero.

Solution:
This follows from the divergence theorem or from Stokes theorem.

5) T **F** The curve $\vec{r}(t) = [t, t, t^2]$ intersects the plane $x + y + 2y = 5$ in a right angle.

Solution:
The velocity vector would have to be perpendicular to the surface. This was actually a typo. I wanted 2z and not 2y in which case it would have been true. As stated it is false.

6) T F The surface $(x-5)^2-(y+1)^2+z^2=-1$ is a two-sheeted hyperboloid.

Solution:
One can see that by putting $y=-1$.

7) T F If $\vec{F}$ is a vector field with $|\vec{F}(x,y,z)|=1$ for all points, and $|S|$ is the surface area of a surface S, then $\iint_S \vec{F} \cdot d\vec{S} \leq |S|$.

Solution:
The maximum we can get if F is perpendicular to S at all points.

8) T F For any unit vector $\vec{v}$ we have $|\vec{v} \cdot [1,0,0]| \leq 1$.

Solution:
By Cauchy Schwarz

9) T F If the acceleration $\vec{r}\,''(t)=0$ at all t, then $\vec{r}(t)$ moves on a straight line.

Solution:
This is one of the observations of Newton. We also accepted the answer False here as one can have that the point is not moving at all.

10) T F For a field $\vec{F}$ and curve $\vec{r}$, we have $\frac{d}{dt}\vec{F}(\vec{r}(t)) = \text{curl}(\vec{F})(\vec{r}(t)) \cdot (\vec{r}(t)) \times \vec{r}\,'(t))$.

Solution:
This is a crazy relation. Essentially any random example of a field shows that it is wrong.

11) T F If $\vec{F}$ and $\vec{G}$ are both gradient fields, then $\vec{F}+\vec{G}$ is a gradient field.

Solution:
The potential is the sum of the potentials.

12) T F $\vec{r}(s,t) = [\cos(t^3)\sin(s^3), \sin(t^3)\sin(s^3), \cos(s^3)]$ parametrizes a sphere.

Solution:
It is a reparametrization.

Solution:
This was a homework.

13) T F The flux of $\vec{F} = [0, 7y, -4z]$ through the sphere $x^2 + y^2 + z^2 = 1$ oriented outwards is equal to 4π.

Solution:
It is the volume by the divergence theorem.

14) T F The set of points satisfying $\phi^2 = 1$ in spherical coordinates is a cone.

Solution:
Yes, $\phi = 1$ gives a single cone. If one includes $\phi = -1$ one gets a double cone.

15) T F There exists a vector field $\vec{F}$ and a scalar function $g(x, y, z)$ such that $\mathrm{curl}(\vec{F}) = \mathrm{grad}(g)$.

Solution:
Take $g = 2z$, $F = [-y, x, 0]$.

16) T F The solid defined by $x^2 + y^2 + z^2 \leq 9, z \leq \sqrt{x^2 + y^2}$ has volume $\int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^3 \rho^2 \sin(\phi) \, d\rho d\phi d\theta$.

Solution:
This is the volume in cylindrical coordinates but the phi angle range is wrong. We have to go from pi/4 to pi

17) T F For any vector field $\vec{F} = [P, Q, R]$, we have $|\mathrm{div}(\vec{F})| \leq |\mathrm{curl}(\vec{F})|$.

Solution:
Any relation is possible. An example showing it to be false is $[x, y, z]$.

18) | T | F | $\text{grad}(\text{div}(\vec{F})) = \text{curl}(\text{curl}(\vec{F}))$.

Solution:
While both sides make sense as expressions, this is wrong for most fields. Take a gradient field for example of the form $[x^4, 0, 0]$. Then the right hand side is zero but the left hand side is not.

19) | T | F | George Green from Green's theorem was an English aristocrat.

Solution:
He was a miller. Certainly not an aristocrat. We showed a bit of a movie from good will hunting and also mentioned Ramanujan in class. This question was here to tease a bit any student who has not payed attention in class during the Green's lecture.

20) | T | F | If $\vec{N}(t)$ is the normal vector in the TNB-frame to a curve $\vec{r}(t)$, then $|\vec{N}\,'(t)|/|\vec{r}\,'(t)|$ is the curvature.

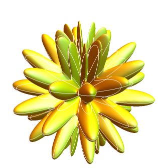
Solution:
We have to replace N with T.

Problem 2) (10 points) No justifications are necessary.

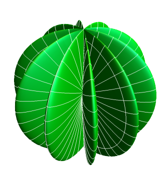
a) (2 points) Match the surfaces given in spherical coordinates. There is an exact match.

Surface	A-C
$\rho = \sin^2(5\phi)$	
$\rho = \sin^2(5\theta)$	
$\rho = \sin^2(5\theta) + \sin^2(5\phi)$	

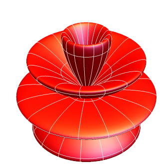
A



B



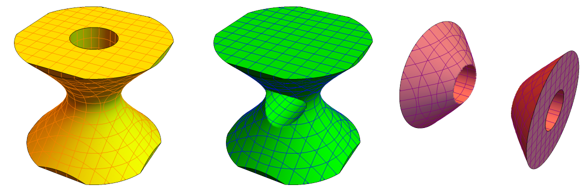
C



b) (2 points) Match the solids. There is an exact match.

Solid	A-C
$x^2+y^2-z^2<2, x^2+z^2>1$	
$x^2+y^2-z^2<2, x^2+y^2>1$	
$x^2-y^2-z^2>1, y^2+z^2>1$	

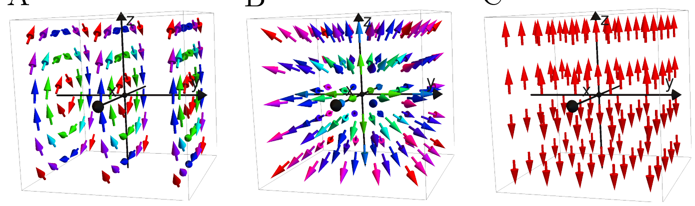
A B C



c) (2 points) The figures display vector fields. There is an exact match.

Field	A-C
$\vec{F}=[0,0,z]$	
$\vec{F}=[x,y,z]$	
$\vec{F}=[-z,0,x]$	

A B C



d) (2 points) Recognize partial differential equations! You saw Klein-Gordon in the homework.

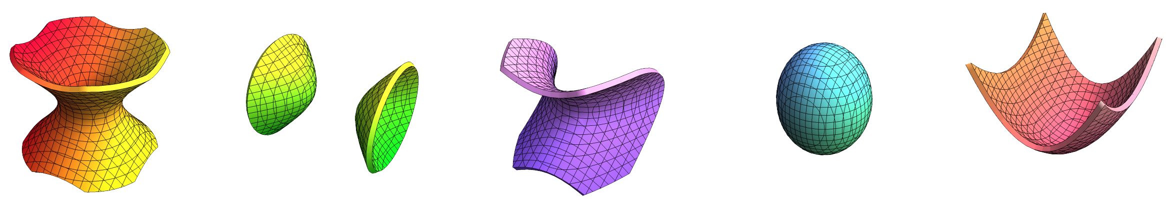
Equation	A-F
Transport	
Burger	
Laplace	
Klein-Gordon	

	PDE
A	$g_T = g_X$
B	$g_{TT} = -g_{XX}$
C	$g_T^2 - 1 = g_X^2$

	PDE
D	$g_T = g_X g_{XX}$
E	$g_{XX} - g_{TT} = g$
F	$g_{XX} + g g_X = g_T$

e) (2 points) Some quadrics

	Enter a letter from A-E each
Pick the one sheeted hyperboloid	
Pick the ellipsoid	
Pick the hyperbolic paraboloid	



A B C D E

Solution:
CBA, BAC, CBA, AFBE, ADC

Problem 3) (10 points) No justifications necessary

Fill in all the numbers 1-10 exactly once. While there could be multiple solutions to an individual question, the match is perfect.

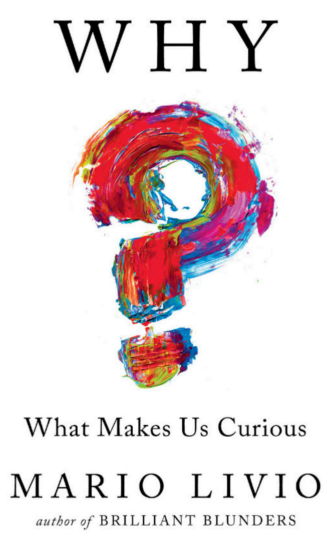
Why?	Because (enter 1-10)
Why are gradients of f and g parallel when maximizing f under g=0?	
Why is div(curl(F)) $= 0$?	
Why is the gradient $\nabla f(x,y)$ zero at maxima or minima of f?	
Why can one change the order of integration for rectangles?	
Why is the line integral of a gradient field along a circle 0?	
Why is the flux of the curl through a sphere zero?	
Why is volume of a solid determined by the boundary?	
Why is the gradient of a function perpendicular to the level curve?	
Why is $\|\vec{v}\cdot\vec{w}\|/(\|\vec{v}\|\|\vec{w}\|) \leq 1$ for any $\vec{v},\vec{w}$?	
Why is the area of a region determined by its boundary?	

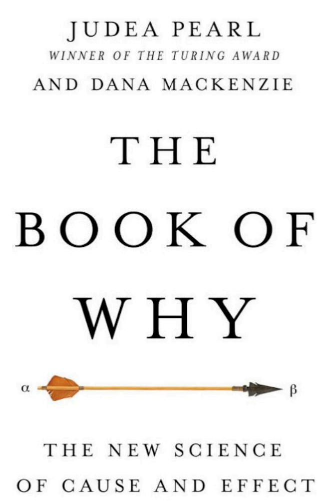
1 Stokes theorem, 2 Chain rule, 3 Clairaut, 4 Cauchy-Schwarz, 5 Green, 6 Fubini, 7 Lagrange, 8 Fermat, 9 Divergence theorem, 10 Fundamental theorem of line integrals.

The question Why is important as the literature displayed to the right indicates.

Curiosity is the motor of science.

Asking good questions is pivotal for understanding.





Solution:
7,3,8,6,10,1,9,2,4,5.

Problem 4) (10 points)

We explore our surroundings with a **small drone** of less than a quarter kilograms. It climbs with an acceleration

$$\vec{r}\,''(t) = [0, 0, 2 + \sin(t)]$$

with initial velocity $\vec{r}\,'(0) = [1, 0, 0]$ and initial position $\vec{r}(0) = [3, 4, 10]$.

a) (6 points) Where is the drone at time $t = 2\pi$?

b) (4 points) What is the **curvature** $\kappa(\vec{r}(0))$ of the curve $\vec{r}(t)$ at $t = 0$?

Solution:
a) Integrate twice to get $\vec{r}\,'(t) = [1, 0, 2t - \cos(t) + 1]$ and $\vec{r}(t) = [t + 3, 4, t^2 + \sin(t) + t + 10]$
At $t = 2\pi$ we are at $[2\pi + 3, 4, 4\pi^2 + 10 + 2\pi]$.
b) We already are given $r'(0) = [1, 0, 0]$ and see also $r''(0) = [0, 0, 2]$. The curvature is $\kappa = |r'(t) \times r''(0)|/|r'(0)|^3 = 2$.

Problem 5) (10 points)

a) (2 points) Find the **tangent plane** to the level surface

$$g(x, y, z) = 2x^2 + y^3 - z = 0$$

at the point $(2, 1, 9)$.

b) (2 points) Remember that the level curve $f(x, y) = 2x^2 + y^3 = 9$ is a curve in the plane. Find the **tangent line** to the point $(2, 1)$.

c) (2 points) Estimate the value of $2 \cdot 2.001^2 + 0.99^3$ using **linearization**.

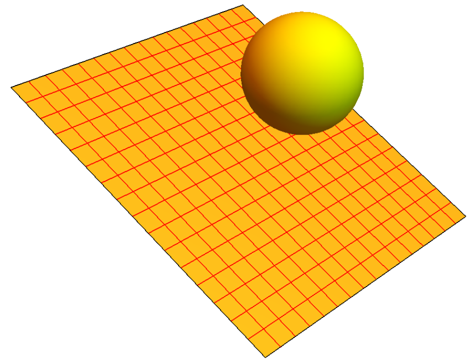
d) (2 points) What is the **directional derivative** $D_{[0,1]}f(2,1)$?

e) (2 points) Check the **implicit differentiation formula** $f_x(2,1) = -g_x(2,1,9)/g_z(2,1,9)$.

Solution:
The key in a),b) is to compute the gradient $\nabla g(2,1,9) = [8,3,-1]$ and $\nabla f(1,2) = [8,3]$ then adjust the constant.
a) $8x + 3y - z = 10$.
b) $8x + 3y = 19$.
c) $9 + 8 * 0.001 - 3 * 0.01 = 8.978$.
d) The second component of the gradient is 3.
e) The left hand side is the first component of the gradient 8. The second is $-8/(-1) = 8$.

Problem 6) (10 points)

a) (4 points) Find the **distance** of the sphere $x^2 + y^2 + z^2 = 1$ to the plane containing the points $A = (4, 0, 0), B = (0, 6, 0), C = (0, 0, 2)$.



b) (3 points) What is the **implicit equation** $ax + by + cz = d$ of the plane containing A, B, C?

c) (3 points) What is the **area** of the triangle ABC?

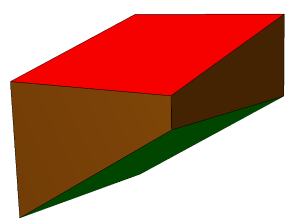
Solution:
a) The distance to the origin is 12/7. The distance to the sphere is 5/7.
b) The vectors $v = [-4, 6, 0]$ and $w = [-4, 0, 2]$ are in the plane. Take the cross product and adjust the constant to get

$$12x + 8y + 24z = 48 .$$

c) The area of the triangle is half the area of the parallelogram which is the length $|v \times w|/2 = 28/2 = 14$.

Problem 7) (10 points)

a) (5 points) A **house** has the **ground** $z = 2y - 2$ and **roof** $z = 2 - x$ and is located above the square $0 \leq x \leq 1, 0 \leq y \leq 1$. Find the volume of the house using a **triple integral**. We want to see the triple integral!



b) (5 points) Integrate the **charge density function** $f(x, y, z) = z^4$ over the solid E given by

$$x^2 + y^2 + z^2 \leq 9, z > 0 .$$

Solution:

a) Use Cartesian coordinates $\int_0^1 \int_0^1 \int_{2y-2}^{2-x} 1 \; dzdydx$ which is $5/2$.

b) This can be done in cylindrical coordinates (polar in x, y). Much simpler is spherical coordinates.

$$\int_0^{2\pi} \int_{\pi/2}^{\pi} \int_0^3 \rho^4 \cos^4(\phi) \rho^2 \sin(\phi) \; d\rho d\phi d\theta$$

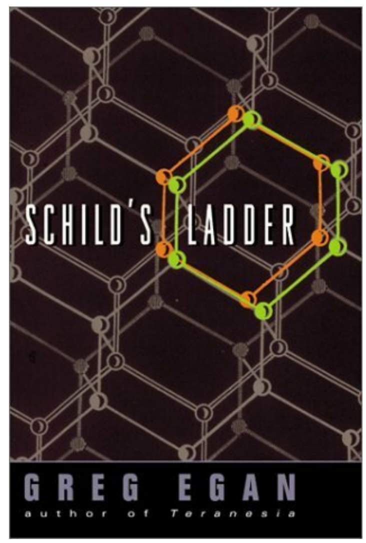
which is $3^7 2\pi/35$.

Problem 8) (10 points)

a) (8 points) While study the **novo vacuum**, we are led to the problem to find the extrema of the function

$$f(x,y) = x^2 - \frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5} + y^2$$

and classify them using the **second derivative test**. The humanoid physicist Cass (a specialist in quantum graph theory) has already figured out that all critical points have integer coordinates and wants you to know this.



b) (2 points) Answer whether there is a global maximum or global minimum for f.

We are motivated by the novel "Schild's ladder" of Greg Egan tells the story of some humanoid physicists who need to escape and at the same time under an expanding **bubble of novo vacuum**, which is more stable than the ordinary vacuum. Mathematically, it is possible that our vacuum is only a local minimum of an energy functional and that there is a global minimum nearby. By the way, Schild's ladder is a tough book to read. It belongs to the category of hard science fiction.

Solution:

Point	D	f_{xx}	nature	f
(x,y)	D	$f_x x$	Type	f
(-1,0)	-12	-6	saddle	19/30
(0, 0)	4	2	minimum	0
(1, 0)	-4	-2	saddle	11/30
(2,0)	12	6	minimum	-4/15

b) there is no global maximum and global minimum because for $y = 0$ already we have a function which has no global minimum nor global maximum.

Problem 9) (10 points)

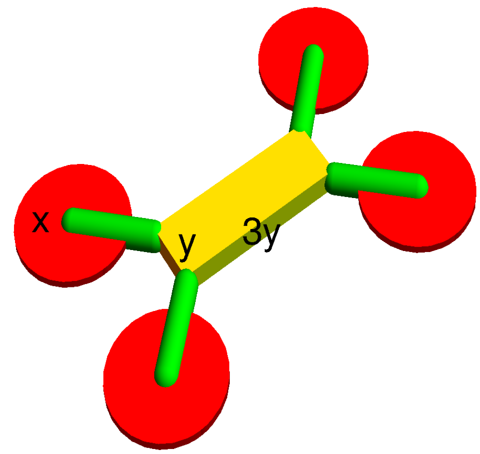
We want to minimize the weight

$$f(x, y) = 4x^2 + 9y^2$$

covered by four propellers of weight x^2 each and body of a **micro drone** with weight $9y^2$ while working with the constaint that the skeleton length

$$g(x, y) = 4x + 8y = 100$$

is constant. You need to solve the problem with the method we have learned in this course, even if you can solve the problem differently. The smallest micro drones one can buy have the size of **dragon flies**. Robotics labs work on drones and robo bees having the size of an actual fly.



Solution:
The Lagrange equations are $8x = \lambda 4, 18y = \lambda 8$. This gives $64x = 72y$ or $8x = 9y$. Plugging into the third equation $4x + 8y = 100$ gives $x = 8, y = 9$. The minimum is $(x, y) = (8, 9)$.

Problem 10) (10 points)

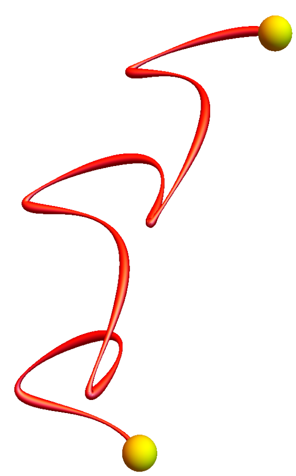
What is the line integral of

$$\vec{F}(x, y, z) = [3x^3 + yz, 3y^3 + xz, 3z^3 + xy]$$

along the curve

$$\vec{r}(t) = [t^2 + 3t + \sin(\sin(\pi 15t)), t^3, t^2 + \sin(\sin(\pi 15t))] ,$$

where $0 \leq t \leq 1$?



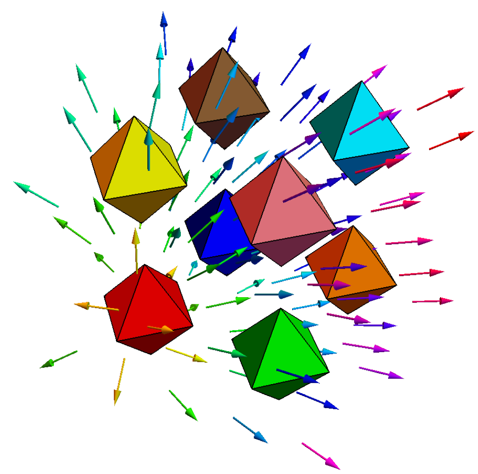
Solution:
To use the fundamental theorem of line integrals we first notice that $\vec{F}$ is a gradient field with potential $f(x, y, z) = 3(x^4 + y^4 + z^4)/4 + xyz$. By the fundamental theorem, the result is $f(4, 1, 1) - f(0, 0, 0) = 395/2$.

Problem 11) (10 points)

As part of an art project **"octogrid of octahedra"**, we radiate a laser field $\vec{F}$ through a solid composed of **8 gems**, each being an octahedron of length 1. Each single octahedron is known to have volume $\sqrt{2}/3$ of course (base area time height $\sqrt{2}$ divided by 3). The radiation field is

$$\vec{F} = [5\sqrt{2}x + \sin(yz), 4\sqrt{2}y + \cos(xz), 3\sqrt{2}z + \sin(xy)] .$$

What is the total flux of $\iint_S \vec{F} \cdot d\vec{S}$ through the boundary surface S of **the union of all these** 8 **gems** if the boundary is oriented outwards on each gem?



Solution:
We use the divergence theorem. The divergence is $12\sqrt{2}$. The result is $8 \cdot (\sqrt{2}/3) \cdot 12\sqrt{2} = 64$.

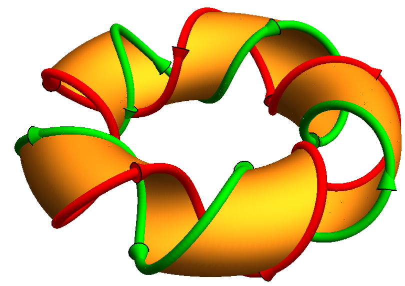
Problem 12) (10 points)

A **ribbon** S is parametrized as

$$\vec{r}(t,s) = [(3+\cos(s))\cos(t), (3+\cos(s))\sin(t), \sin(s)] ,$$

with two parameters

$$0 \le t \le 2\pi, 5t - \pi \le s \le 5t .$$



It is bounded by the curves

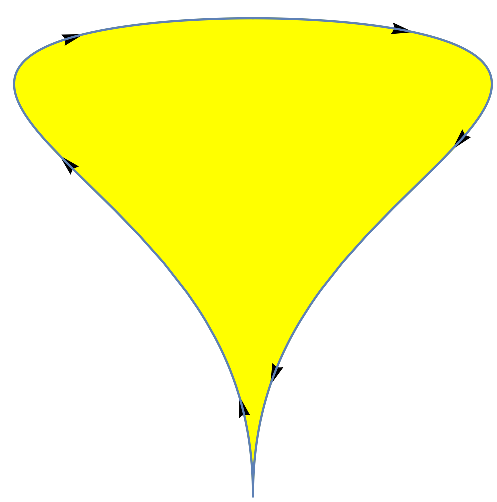
$$\vec{r}_1(t) = [\cos(t)(3+\cos(5t)), \sin(t)(3+\cos(5t)), \sin(5t)]$$

$$\vec{r}_2(t) = [\cos(t)(3-\cos(5t)), \sin(t)(3-\cos(5t)), -\sin(5t)] ,$$

where t goes from 0 to 2π. The orientation of these curves (counter clockwise when looking from above) is indicated in the picture. In each case, it might or might not be compatible with the orientation of the outwards oriented ribbon S. Compute the flux of the curl of $\vec{F}(x,y,z) = [0, 0, \sqrt{x^2+y^2} - 3]$ through S.

Solution:
To use Stokes theorem, we need $\vec{F}(\vec{r}_1(t)) = [0, 0, \cos(5t)]$ and $\vec{F}(\vec{r}_2(t)) = [0, 0, -\cos(5t)]$ and also need $\vec{r}_1\,'(t) = [..., ..., 5\cos(5t)]$ and $\vec{r}_2\,'(t) = [..., ..., -5\cos(5t)]$. The line integral along the first path is $\int_0^{2\pi} [0, 0, \cos(5t)] \cdot [..., ..., 5\cos(5t)] = \int_0^{2\pi} 5\cos^2(t)\, dt = 5\pi$. The line integral along the second path is $\int_0^{2\pi} 5\cos^2(t)\, dt = 5\pi$ again. But the second path is oriented the wrong way. The result is $5\pi - 5\pi = 0$.

Problem 13) (10 points)



While eating breakfast we doodle around with some honey spread on our buttered bread. Find the area of the happily created **honey region** enclosed by

$$\vec{r}(t) = [\frac{\sin^2(\pi t)}{t}, 4 - 4t^2]$$

for $-1 \le t \le 1$. The curve has been traced in the picture.

Solution:
We use Greens theorem with $\vec{F}(x,y) = [0,x]$. We have to compute the line integral

$$\int_{-1}^{1} [0, \frac{\sin^2(\pi t)}{t}] \cdot -8t] \; dt = \int_{-1}^{1} -8\sin^2(\pi t) \; dt = -8 \; .$$

However, as the curve has the wrong orientation, we have to change the sign. The answer is 8.