MULTIVARIABLE CALCULUS

MATH S-21A

Unit 2: Vectors and dot product

Lecture

2.1. Two points $P = (a, b, c)$ and $Q = (x, y, z)$ in space $\mathbf{R}^3$ define a **vector** $\vec{v} = \begin{bmatrix} x-a \\ y-b \\ z-c \end{bmatrix}$. We write this column vector also as a row vector $[x-a, y-b, z-c]$ for typography reasons. The vector starts at P and ends at Q. We write $\vec{v} = \vec{PQ}$. The numbers p, q, r in $\vec{v} = [p, q, r]$ are the **components** of $\vec{v}$. [1]

2.2. Vectors can be attached to any point in space. Two vectors with the same components are considered **equal** as they can be translated into each other. If a vector $\vec{v}$ starts at the origin $O = (0, 0, 0)$, then $\vec{v} = [p, q, r]$ heads from O to the point $P = (p, q, r)$. One can therefore identify **points** $P = (a, b, c)$ with **vectors** $\vec{v} = [a, b, c]$ attached to the origin. For clarity reasons, we often draw an arrow $\vec{\ }$ on top of a vector variable and if $\vec{v} = \vec{PQ}$ then P is the "tail" and Q is the "head" of the vector. To distinguish vectors from points, it is custom to write $[2, 3, 4]$ for vectors and $(2, 3, 4)$ for points. [2]

2.3.

Definition: The **sum** of two vectors is $\vec{u}+\vec{v} = [u_1, u_2]+[v_1, v_2] = [u_1+v_1, u_2+v_2]$. The **scalar multiple** is $\lambda\vec{u} = \lambda[u_1, u_2] = [\lambda u_1, \lambda u_2]$. The **difference** $\vec{u} - \vec{v}$ can best be seen as the addition of $\vec{u}$ and $(-1) \cdot \vec{v}$.

Commutativity, **associativity**, or **distributivity** rules for vectors are inherited from the corresponding rules for numbers. Please review these rules yourself if necessary.

2.4. The vectors $\vec{i} = [1, 0, 0]$, $\vec{j} = [0, 1, 0]$, $\vec{k} = [0, 0, 1]$ are called **standard basis vectors**. This has historically grown because the dot and cross product have grown from **quaternions** which are points (t, x, y, z) in $\mathbb{R}^4$, usually written as $q = t + ix + jy + kz$.

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[1] This notation is compatible with linear algebra course. Stewart notation $\langle p, q, r\rangle$ is not used anywhere else in math.

[2] We avoid the Stewart notation $\langle p, q, r\rangle$, which is harder to read. Programming languages like Python use $[p, q, r]$. Mathematica, Perl or C use $\{p, q, r\}$. In linear algebra it is useful to distinguish row and column vectors.

2.5.

Definition: The **length** $|\vec{v}|$ of a vector $\vec{v} = \vec{PQ}$ is defined as the distance $d(P, Q)$ from P to Q. A vector of length 1 is called a **unit vector**. If $\vec{v} \neq \vec{0}$, then $\vec{v}/|\vec{v}|$ is called a **direction** of $\vec{v}$. The only vector of length 0 is the zero vector $\vec{0} = [0, 0, 0]$.

2.6.

Definition: The **dot product** of two vectors $\vec{v} = [a, b, c]$ and $\vec{w} = [p, q, r]$ is defined as $\vec{v} \cdot \vec{w} = ap + bq + cr$.

2.7. Different notations for the dot product are used in different mathematical fields. While mathematicians write $\vec{v} \cdot \vec{w}$ or $(\vec{v}, \vec{w})$ or $\langle \vec{v}, \vec{w} \rangle$, the **Dirac notation** $\langle \vec{v} | \vec{w} \rangle$ is used in quantum mechanics. The **Einstein notation** $v_i w^i$ or more generally $g_{ij} v^i w^j$ is used in general relativity. In statistics, it is called the **covariance** $\text{Cov}[v, w]$ of centered data. The dot product is also called **scalar product** or **inner product**. It could be generalized. Any product $g(v, w)$ which is bi-linear (linear both in v and w) and satisfies the symmetry $g(v, w) = g(w, v)$ and $g(v, v) \geq 0$ and $g(v, v) = 0$ if and only if $v = 0$, can be used as a dot product. An example is $g(v, w) = 2v_1 w_1 + 3v_2 w_2 + 5v_3 w_3$.

2.8. The dot product determines distances. A bit surprising, distances determine the dot product: **Proof:** Express the length of $\vec{v}$ as $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$. On the other hand, from $(\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2(\vec{v} \cdot \vec{w})$ can be solved for $\vec{v} \cdot \vec{w}$:

$$\vec{v} \cdot \vec{w} = (|\vec{v} + \vec{w}|^2 - |\vec{v}|^2 - |\vec{w}|^2)/2 \ .$$

2.9. The **Cauchy-Schwarz inequality** is

Theorem: $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$.

Proof. If $|\vec{w}| = 0$, the statement holds because both sides are zero. Otherwise, assume $|\vec{w}| = 1$ by dividing the equation by $|\vec{w}|$. Now plug in $a = \vec{v} \cdot \vec{w}$ into the equation $0 \leq (\vec{v} - a\vec{w}) \cdot (\vec{v} - a\vec{w})$ to get $0 \leq (\vec{v} - (\vec{v} \cdot \vec{w})\vec{w}) \cdot (\vec{v} - (\vec{v} \cdot \vec{w})\vec{w}) = |\vec{v}|^2 + (\vec{v} \cdot \vec{w})^2 - 2(\vec{v} \cdot \vec{w})^2 = |\vec{v}|^2 - (\vec{v} \cdot \vec{w})^2$ which means $(\vec{v} \cdot \vec{w})^2 \leq |\vec{v}|^2$. $\square$

2.10. Having established this, it is possible to give, without referring to geometric pictures, a definition of what an **angle** is:

Definition: The **angle** between two nonzero vectors $\vec{v}, \vec{w}$ is defined as the unique $\alpha \in [0, \pi]$ which satisfies $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$. Since cos maps $[0, \pi]$ in a 1:1 manner to $[-1, 1]$, this is well defined.

2.11. The **Al Kashi theorem** gives the third side length c of a triangle ABC in terms of the sides $a = d(A, C), b = d(B, C)$ and α, the angle located at the vertex A.

Theorem: $a^2 + b^2 = c^2 - 2ab\cos(\alpha)$.

Proof. Define $\vec{v} = \vec{AB}, \vec{w} = \vec{AC}$. Because $c^2 = |\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{v} \cdot \vec{w}$, We know $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$ so that $c^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}| \cdot |\vec{w}| \cos(\alpha) = a^2 + b^2 - 2ab\cos(\alpha)$. $\square$

2.12. The **triangle inequality** tells

> **Theorem:** $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$

Proof. $|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) = \vec{u}^2 + \vec{v}^2 + 2\vec{u} \cdot \vec{v} \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u} \cdot \vec{v}| \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u}| \cdot |\vec{v}| = (|\vec{u}| + |\vec{v}|)^2$. $\square$

> **Definition:** Two vectors are called **orthogonal** or **perpendicular** if $\vec{v} \cdot \vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. For example, $\vec{v} = [2, 3]$ is orthogonal to $\vec{w} = [-3, 2]$.

2.13. We can now prove the **Pythagorean theorem**:

> **Theorem:** If $\vec{v}$ and $\vec{w}$ are orthogonal, then $|v - w|^2 = |v|^2 + |w|^2$.

Proof. $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w}$. We usually write this as $a^2 + b^2 = c^2$. $\square$

2.14.

> **Definition:** The vector $\mathrm{P}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \vec{w}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$. The **scalar projection** $\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|}$ is a **signed length** of the vector projection. Its absolute value is the length of the projection of $\vec{v}$ onto $\vec{w}$. The vector $\vec{b} = \vec{v} - P(\vec{v})$ is a vector orthogonal to the $\vec{w}$-direction.

2.15. The projection allows to **visualize** the dot product. The absolute value of the dot product is the length of the projection. The dot product is positive if $\vec{v}$ points more towards to $\vec{w}$, it is negative if $\vec{v}$ points away from it. In the next class, we use the projection to compute distances between various objects.

Examples

2.16. For example, with $\vec{v} = [0, -1, 1]$, $\vec{w} = [1, -1, 0]$, $\mathrm{P}(\vec{v}) = [1/2, -1/2, 0]$. Its length is $1/\sqrt{2}$.

2.17. The RGB **color space** consists of triples $\vec{v} = [r, g, b]$ describing the amount of red, green and blue of a **color**. An other coordinate system is the CMY **color space** consisting of triples $\vec{v} = [c, m, y] = [1 - r, 1 - g, 1 - b]$, where c is **cyan**, m is **magenta** and y is **yellow**.

2.18. In physics, forces and fields $\vec{F}$ are described by vectors. The **velocity** of a curve $r(t) = [x(t), y(t), z(t)]$ is a vector attached to the point $r(t)$. We will come to this.

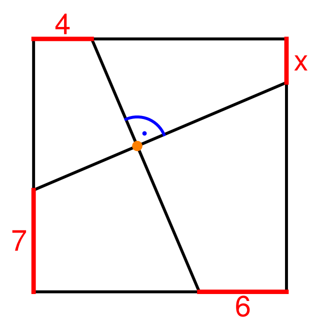
2.19. In probability theory, data are described by vectors. One calls them also **random variables**. It is in statistics, where higher dimensional spaces appear. In statistics, $\vec{v} \cdot \vec{w}$ is the **covariance** and $\cos(\alpha) = (\vec{v} \cdot \vec{w})/(|\vec{v}||\vec{w}|)$ is the **correlation**.

Homework

This homework is due on Tuesday, 7/01/2025.

Problem 2.1: a) Find a **unit vector** parallel to $\vec{x} = \vec{u} + \vec{v} + \vec{w}$ if $\vec{u} = [4, 3, 1]$ and $\vec{v} = [2, 4, 1]$ and $\vec{w} = [0, 0, 4]$.
b) Verify that in a right angle triangle, the relation $1/a^2 + 1/b^2 = 1/h^2$ holds, where h is the height perpendicular to the hypotenuse c.

Problem 2.2: Find x in the following picture about a square. The riddle appeared in a German magazine. Please document how you think about it and also record also attempts which failed.



Problem 2.3: **Colors** are encoded by vectors $\vec{v} = [$ red , green , blue $]$. The red, green and blue components of $\vec{v}$ are all real numbers in the interval $[0, 1]$.
a) Determine the angle between the colors yellow and magenta.
b) What is the vector projection of the magenta-orange mixture $\vec{x} = (\vec{v} + \vec{w})/2$ onto green $\vec{y}$?

Problem 2.4: A rope is wound exactly 8 times around a stick of circumference 1 and length 15. How long is the rope?

Problem 2.5: a) Find the angle between the main diagonal of the unit cube and one of the face diagonals. Assume that both diagonals pass through a common vertex.
b) Find the vector projection of the main diagonal $\vec{v} = [1, 1, 1]$ onto the side diagonal $\vec{w} = [1, 1, 0]$.
c) Find the maximal distance between the 16 points $(\pm 1, \pm 1, \pm 1, \pm 1)$ of a **tesseract**.