MULTIVARIABLE CALCULUS

MATH S-21A

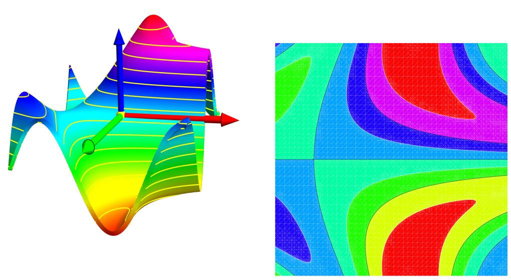
Unit 5: Functions

Lecture

5.1. A function f of two variables assigns a scalar numerical quantity $f(x,y)$ to a point (x,y) in the plane. It could be a temperature for example. If $f(x,y)$ is drawn in the third dimension, we get a surface called the **graph** of f.

Definition: A **function of two variables** $f(x,y)$ is a rule which assigns to two real numbers x,y a third real number $f(x,y)$.

The function $f(x,y) = x^3y - \log(1+x^2y^2)$ for example assigns to $(1,2)$ the number $f(1,2) = 2 + \log(5)$ and to $(0,0)$ the number $f(0,0) = 0$.



5.2. Usually, a function is defined for all points (x,y) in $\mathbb{R}^2$ like in the case $f(x,y) = |x|^7 + e^{\sin(xy)}$. Sometimes however it is required or desired to restrict the function to a **domain** D. For example, if $f(x,y) = \log|y| + \sqrt{x}$, then (x,y) is only defined for $x > 0$ and for $y \neq 0$. An other example is when x,y are non-negative quantities like length. The **range** of a function f is the set of all possible values which the function f reaches. The function $f(x,y) = 3 + x^2/(1+x^2)$ for example takes all values $\{3 \leq z < 4\}$. While $z = 3$ is reached for $x = y = 0$, and all values $z < 4$ are obtained, the value $z = 4$ is not in the range. This is why we wrote $z < 4$.

5.3.

Definition: The set $\{(x,y,f(x,y)) \mid (x,y) \in D\} \subset \mathbb{R}^3$ is called the **graph** of f.

5.4. Graphs are **surfaces** that visualize the function. We do not want to mix up the graph of a function with the function itself. The function is a **rule** which assigns to (x, y) a third number, the graph is a **geometric object** in three dimensional space. The modern notion of function started to appear at the beginning of the 19'th century. That the function f and the graph of f are different can matters in computer science. They are implemented as different **data structures**. In Mathematica, $f = Function[x, y, Sin[xy]]$ defines a function which we can evaluate like $f[0, 0]$, but its graph $S = Plot3D[f[x, y], x, -1, 1, y, -1, 1]$ is a geometric graphics object. In some cases, we know f but we do not understand the graph. An example is the **zeta function** $f(x, y) = |\zeta(x + iy)|$ for which we know the function very well, can evaluate it at every point but where we do not know the graph, especially, where its zeros are. A million dollars wait for whom can show that no root happens for $x > 1/2$ or $x < 1/2$ or point out a root (x, y) with $0 < x < 1/2$.

5.5. Here are some examples of functions

$f(x,y)$	domain D of f	range $= f(D)$ of f
$\sqrt{9 - x^2 - y^2}$	closed disc $x^2 + y^2 \leq 9$	$[0, 3]$
$-\log(1 - x^2 - y^2)$	open unit disc $x^2 + y^2 < 1$	$(0, \infty)$
$f(x, y) = x^2 + y^3 - xy + \cos(xy)$	plane R^2	the real line
$\sqrt{4 - x^2 - 2y^2}$	$x^2 + 2y^2 \leq 4$	$[0, 2]$
$1/(x^2 + y^2 - 1)$	all except unit circle	$R \setminus (-1, 0]$
$1/(x^2 + y^2)^2$	all except origin	positive real axis

5.6.

Definition: The set $\{(x, y) \mid f(x, y) = c = \text{const}\ \}$ is called a **contour curve** or **level curve** of f. Note that it can be empty, consist of points only or be the entire domain like for $f(x, y) = 0$ and $c = 0$. A collection of contour curves is called a **contour map**.

For $f(x, y) = 4x^2 + 3y^2$ for example, the level curves $f = c$ are **ellipses** if $c > 0$. Drawing several contour curves $\{f(x, y) = c\ \}$ simultaneously produces a **contour map** of the function f.

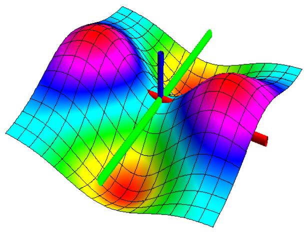
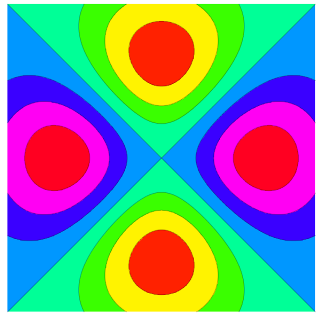
5.7. Level curves allow to visualize and analyze functions $f(x, y)$ without having to leave the 2-dimensional space. The picture below for example shows the level curves of the function $\sin(xy) - \sin(x^2 + y)$. Contour curves are everywhere: they appear as **isobars**=curves of constant pressure, or **isoclines**= curves of constant (wind) field direction, **isothermes**= curves of constant temperature or **isoheights** =curves of constant height.

Examples

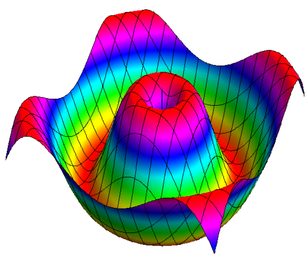
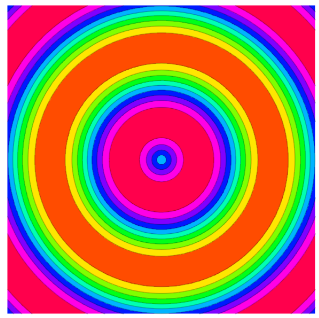
5.8. For $f(x, y) = x^2 - y^2$, the set $x^2 - y^2 = 0$ is the union of the lines $x = y$ and $x = -y$. The set $x^2 - y^2 = 1$ consists of two hyperbola with with their "noses" at the point $(-1, 0)$ and $(1, 0)$. The set $x^2 - y^2 = -1$ consists of two hyperbola with their noses at $(0, 1)$ and $(0, -1)$.

5.9. The function $f(x, y) = 1 - 2x^2 - y^2$ has contour curves $f(x, y) = 1 - 2x^2 - y^2 = c$ which are ellipses $2x^2 + y^2 = 1 - c$ for $c < 1$.

5.10. For the function $f(x,y) = (x^2 - y^2)e^{-x^2-y^2}$, we can not find explicit expressions for the contour curves $(x^2-y^2)e^{-x^2-y^2} = c$. We can draw the traces curve $(x, 0, f(x, 0))$ or $(0, y, f(0, y)$ or then use a computer:

5.11. The surface $z = f(x,y) = \sin(\sqrt{x^2 + y^2})$ has concentric circles as contour curves.

5.12. In applications, discontinuous functions can occur. The temperature of water in relation to pressure and volume is an example. One experiences then **phase transitions**, places where the function value can jumps. Mathematicians study singularities in a mathematical field called "catastrophe theory".

5.13.

Definition: A function $f(x,y)$ is called **continuous** at (a,b) if there is a finite value $f(a,b)$ with $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. This means that for any sequence (x_n, y_n) converging to (a,b), also $f(x_n, y_n) \to f(a,b)$. A function is **continuous** in $G \subset \mathbb{R}^2$ if it is continuous at every point (a,b) of G.

5.14. Continuity means that if (x,y) is close to (a,b), then $f(x,y)$ must be close to $f(a,b)$. Continuity for functions of more than two variables is defined in the same way. The bad news is that continuity is not always easy to check. The good news is that in general we do not have to worry about continuity. Lets look at some examples:

5.15. Example: For $f(x,y) = (xy)/(x^2 + y^2)$, we have

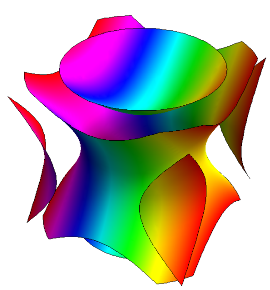
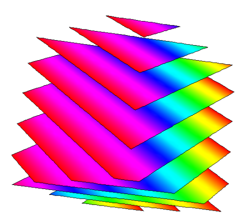
$$\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

and $\lim_{x\to 0} f(0,x) = \lim_{(x,0)\to(0,0)} 0 = 0$. The function is not continuous at $(0,0)$.

5.16. Example: The function $f(x,y) = (x^2y)/(x^2+y^2)$ is better described in polar coordinates: $f(r,\theta) = r^3\cos^2(\theta)\sin(\theta)/r^2 = r\cos^2(\theta)\sin(\theta)$. We see that $f(r,\theta) \to 0$ uniformly in θ if $r \to 0$. The function is continuous as we can extend it and extend the value to $f(0,0) = 0$. It is custom in mathematics to consider the above function **to be continuous**. The reason is that there is a **unique way** to give a function value at the undefined point.

5.17. A simpler example: the function $f(x,y) = (x^2-y^2)/(x+y)$ is continuous everywhere. Yes, the function is not defined a priori at $x+y=0$ but as it is outside this line equal to $f(x,y) = x-y$, there is a unique continuation to the entire plane and this continuation is $x-y$.

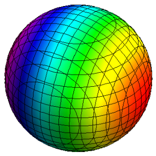
5.18. A function of three variables $g(x,y,z)$ assigns to three variables x,y,z a real number $g(x,y,z)$. The function $f(x,y,z) = x^2+y-z$ for example satisfies $f(3,2,1) = 10$. We can visualize a function by **contour surfaces** $g(x,y,z) = c$, where c is constant. It is an **implicit description** of the surface. The contour surface of $g(x,y,z) = x^2+y^2+z^2 = c$ is a sphere if $c > 0$. To understand a contour surface, it is helpful to look at the **traces**, the intersections of the surfaces with the coordinate planes $x=0, y=0$ or $z=0$.

5.19. The function $g(x,y,z) = 2+\sin(xyz)$ could define a temperature distribution in space. We can no more draw the graph of g because that would be an object in 4 dimensions. We can however draw **level surfaces** like $g(x,y,z) = 0$ or $g(x,y,z) = 1$.

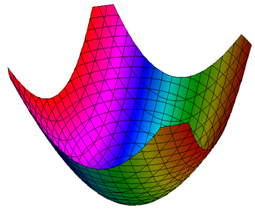
5.20. The level surfaces of $g(x,y,z) = x^2+y^2+z^2$ are spheres. The level surfaces of $g(x,y,z) = 2x^2+y^2+3z^2$ are ellipsoids. The equation $ax+by+cz = d$ is a plane. With $\vec{n} = [a,b,c]$ and $\vec{x} = [x,y,z]$, we can rewrite the equation $\vec{n}\cdot\vec{x} = d$. If a point $\vec{x}_0$ is on the plane, then $\vec{n}\cdot\vec{x}_0 = d$. so that $\vec{n}\cdot(\vec{x}-\vec{x}_0) = 0$. This means that every vector $\vec{x}-\vec{x}_0$ in the plane is orthogonal to $\vec{n}$. For $f(x,y,z) = ax^2+by^2+cz^2+dxy+exz+fyz+gx+hy+kz+m$ the surface $f(x,y,z) = 0$ is called a **quadric**.

Sphere



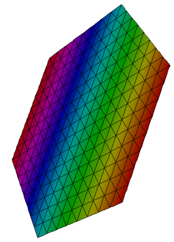
$(x-a)^2+(y-b)^2+(z-c)^2=r^2$

Paraboloid



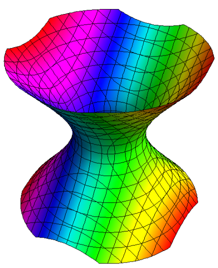
$(x-a)^2+(y-b)^2-c=z$

Plane



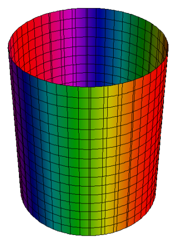
$ax+by+cz=d$

One sheeted hyperboloid



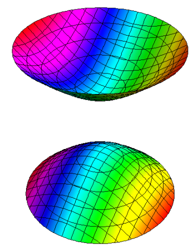
$(x-a)^2+(y-b)^2-(z-c)^2=r^2$

Cylinder



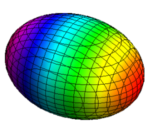
$(x-a)^2+(y-b)^2=r^2$

Two sheeted hyperboloid



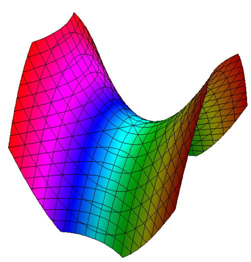
$(x-a)^2+(y-b)^2-(z-c)^2=-r^2$

Ellipsoid



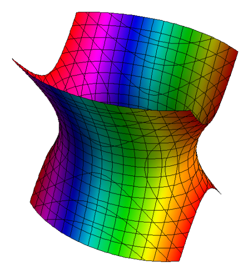
$x^2/a^2+y^2/b^2+z^2/c^2=1$

Hyperbolic paraboloid



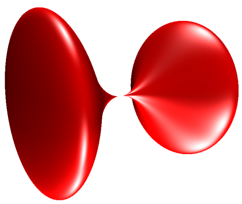
$x^2-y^2+z=1$

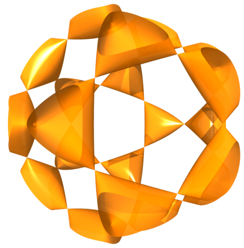
Elliptic hyperboloid

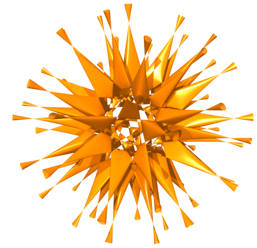


$x^2/a^2+y^2/b^2-z^2/c^2=1$

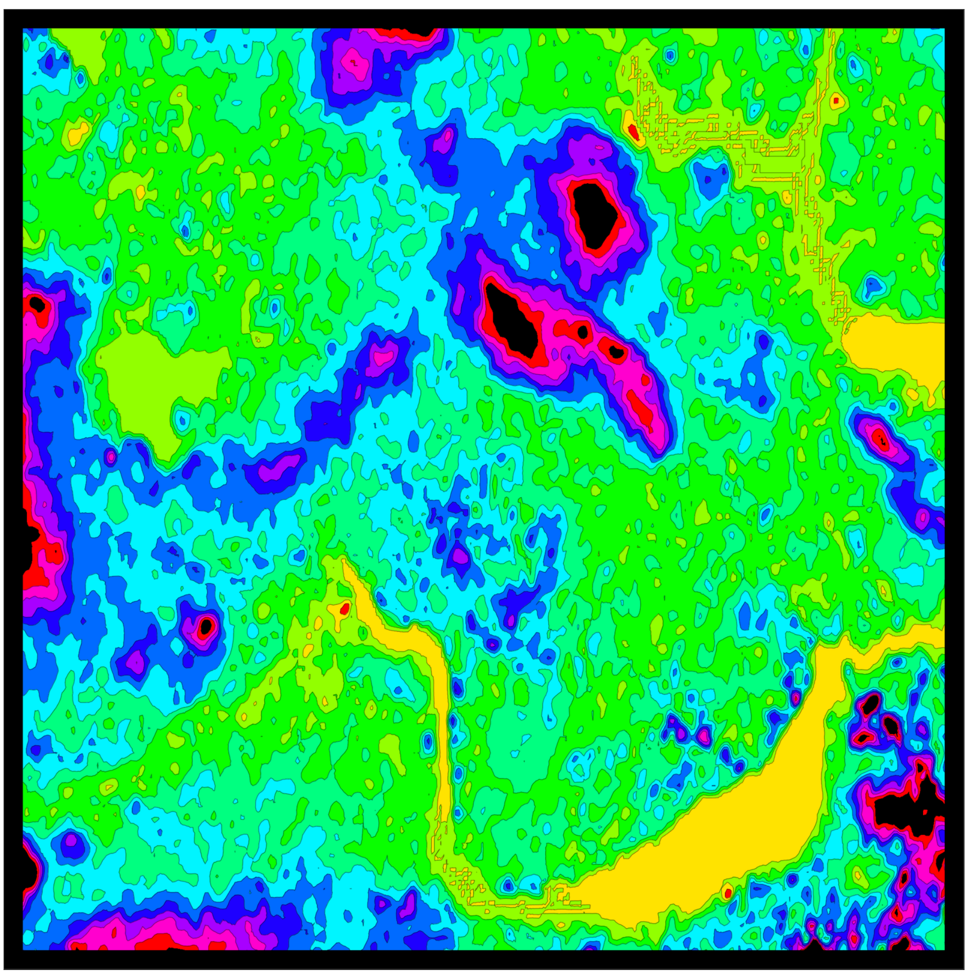
Higher order polynomial surfaces can be intriguingly beautiful and are sometimes difficult to describe. If f is a polynomial in several variables and $f(x,x,x)$ is a polynomial of degree d, then f is called a **degree d polynomial surface**. Degree 2 surfaces are **quadrics**, degree 3 surfaces **cubics**, degree 4 surfaces **quartics**, degree 5 surfaces **quintics**, degree 10 surfaces **decics** and so on.





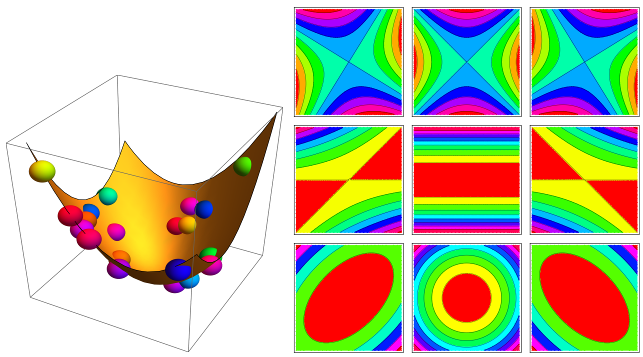


5.21. Contour maps are everywhere. Here is the contour map of the height function near Cambridge.



5.22. Finally, let us mention an interesting challenge problem for an AI. Assume you are given a picture of a surface or a picture of a contour map. Can you find a function $f(x, y)$ which produces this picture? There are various approaches for **machine learning**. [1].

1) Data fitting. In a **data fitting approach** we assume that the function is in a known class like polynomials. Now we just give a few data points and then find the best fit.



Here is the Mathematica code which does this for 20 data points. The unknown function is $f(x, y) = x^2 + y^2 + xy/2$.

```
R:=2Random[]-1;data=Table[u=R;v=R;{u,v,u^2+v^2+u*v/2},{20}];
f=Fit[data,{1,x,x^2,y,y^2,x*y},{x,y}];
point[X_]:={Hue[Random[]],Sphere[X,0.14]};
S1=Graphics3D[Map[point,data],AspectRatio->1];
S2=Plot3D[f,{x,-1,1},{y,-1,1},Mesh->False];   Show[{S1,S2}]
```

2) Large language. In a **machine learning approach** , the computer would first generate a lot of pictures and learn what function parts produces what features.

Here is the Mathematica code, with which the machine learns 9 different functions. Given now a contour map, the machine would pick the one which is closest.

```
f[a_,b_]:=a x^2+ b x y + y^2;
GraphicsGrid[Table[ContourPlot[f[a,b],{x,-3,3},{y,-3,3},
   ColorFunction->Hue],{a,-1,1},{b,-1,1}]]
```

3) True intelligence. Discover features and use this to recognize things.

We humans can do that quite well. For example, we see that if $f(x, y)$ does not depend on x, then the contour curves are horizontal lines. Or that if $f(x, y)$ does not depend on y, then the contour curves are vertical lines. Or we see that if the function depends only on $x^2 + y^2$, then the level curves are all circles.

[1] The problem is described in this video *https://www.youtube.com/watch?v=gI1upYgTjf0*

Homework

Due on Tuesday, 7/8/2025

Problem 5.1: Sketch the contour plot and graph of the function $f(x,y) = \exp(-(x-3)^2 - y^2)((x-3)^2 + y^2)$. It is possible to do this without a computer but you can definitely use a computer to check.

Problem 5.2: What is the domain and range of the **logarithmic mean**

$$f(x,y) = \frac{(y-x)}{\log(y) - \log(x)}$$

where log the natural logarithm?
The function is not defined at $x = y$ but one can define $f(x,y)$ on the diagonal $x = y$. Use Hôpital, show that for any b, the limit $\lim_{x \to b} f(x,b)$ exists.

Problem 5.3: Draw the contour surface $((x^2+y^2)^2 - x^2 - y^2)^2 + z^2 = 0.03$ with with x, y, z all in the interval $[-1.3, 1.3]$

Problem 5.4: Draw the Taxi-metric hyperboloid $|x| + |y| - |z| = -1$. You can do that by looking what happens for fixed z and put the pieces together like a 3D printer. You are certainly also allowed to use a computer to draw it.

Problem 5.5: Build a concrete function $f(x,y,z)$ of three variables such that some level surface $f(x,y,z) = c$ is a **pretzel**, a surface with three holes. Hint: the surface $g * h = 0$ is the union of the surfaces $g = 0$ and $h = 0$. Now, $g * h = c$ can produce surfaces in which things are glued nicely. If you should look up a surface on the web or literature, you have to give the reference. You can use the computer to experiment, or then describe your strategy in words.

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