

MULTIVARIABLE CALCULUS

MATH S-21A

Unit 6: Parametrized Surfaces

LECTURE

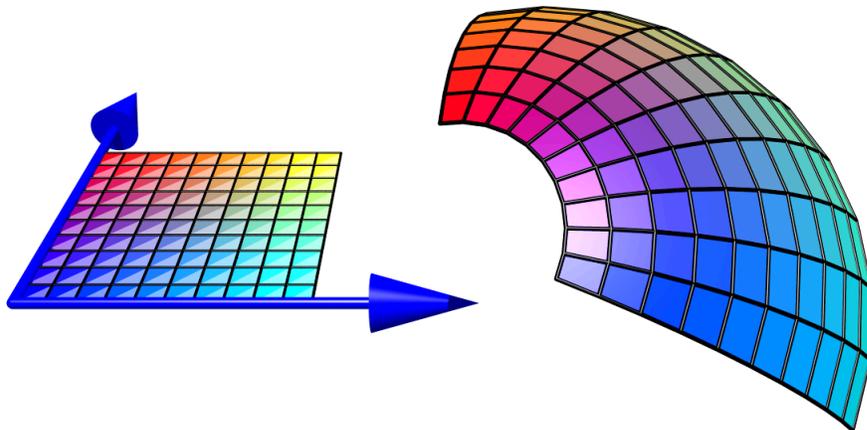
6.1. There are two fundamentally different ways to describe a surface: there is the **level surface** description given by the implicit equation $g(x, y, z) = c$ and there are **parametrizations**. We have seen that for planes already. Lets look at the general setup.

Definition: A **parametrization** of a surface is a vector-valued function

$$\vec{r}(u, v) = [x(u, v), y(u, v), z(u, v)] ,$$

where $x(u, v), y(u, v), z(u, v)$ are three functions of two variables. The **parameters** u, v serve as **coordinates** on the surface. For example, if we plug in concrete values like $u = 3, v = 2$ in $\vec{r}(u, v) = [u - 2, v^2, u^3 - v]$, we get a concrete point $\vec{r}(3, 2) = [1, 4, 25]$ in \mathbb{R}^3 .

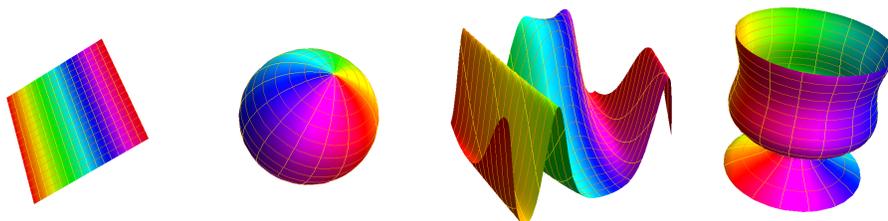
6.2. Because the two parameters are called u and v , the map \vec{r} is also named **uv -map**.
¹ A parametrization is a map from \mathbb{R}^2 to \mathbb{R}^3 . A **parametrized surface** is the image of the uv -map. The domain R of the uv -map is called the **parameter domain**. The parametrization is what you are **doing**, the surface itself is something you **see**. There are many different parametrizations of the same surface.



¹Like uv -light, which also looks cool.

Definition: If the first parameter u is kept constant, then $v \mapsto \vec{r}(u, v)$ is a curve on the surface. Similarly, if v is constant, then $u \mapsto \vec{r}(u, v)$ traces a curve the surface. These curves are called **grid curves**.

6.3. Complicated surfaces can be explored with the help of a computer. The following four examples are important building blocks for more general surfaces.



Definition: A point $(x, y) \neq (0, 0)$ in the plane has the **polar coordinates** $r = \sqrt{x^2 + y^2}, \theta$, where θ is the angle from the positive x -axes to the point in counter clockwise direction. For $x > 0, y > 0$ it is $\arctan(y/x)$. In general $(x, y) = (r \cos(\theta), r \sin(\theta))$. A common choice is to take $\theta \in [0, 2\pi)$. The point $((0, -1)$ has then the polar coordinates $(r, \theta) = (1, 3\pi/2)$.

6.4. $\theta = \arctan(y/x)$ only defines θ up to an integer multiple of π . The points $(1, 2)$ and $(-1, -2)$ for example have the same $\arctan(y/x)$ value. In order to get the correct θ value in $[0, 2\pi)$, one can could just add π if $x < 0$ or if $x = 0$ and $y < 0$. More convenient is to define the angle as $\arg(x + iy)$ where $x + iy \in \mathbb{C}$ is called a complex number. Mathematica for example gives the polar angle with $Arg[x + I * y]$. The letter $i = \sqrt{-1}$ (Mathematica uses I) stands for the imaginary unit. Think of $x + iy \in \mathbb{C}$ as the point $(x, y) \in \mathbb{R}^2$.

Definition: When representing points in space as

$$(x, y, z) = (r \cos(\theta), r \sin(\theta), z)$$

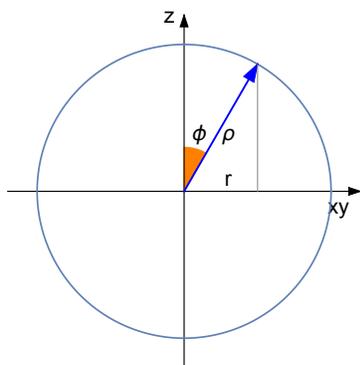
we also use the name **cylindrical coordinates**.

Definition: **Spherical coordinates** use the distance ρ to the origin as well as two angles θ and ϕ called **Euler angles**. The first angle θ is the angle we have used in polar coordinates. The second angle, ϕ , is the angle between the vector \vec{OP} and the z -axis. A point has the **spherical coordinate**

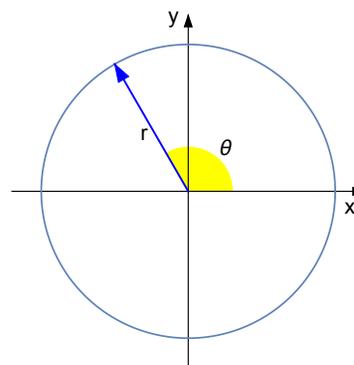
$$(x, y, z) = (\rho \cos(\theta) \sin(\phi), \rho \sin(\theta) \sin(\phi), \rho \cos(\phi)) .$$

We always use $0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi, \rho \geq 0$.

6.5. The following figures allow you to derive the formulas. The distance to the z axes is $r = \rho \sin(\phi)$ and the height $z = \rho \cos(\phi)$ can be read off by the left picture, the coordinates $x = r \cos(\theta), y = r \sin(\theta)$ can be seen in the right picture.



$$\begin{aligned}x &= \rho \cos(\theta) \sin(\phi), \\y &= \rho \sin(\theta) \sin(\phi), \\z &= \rho \cos(\phi)\end{aligned}$$



EXAMPLES

6.6. A plane has the parametrization $\vec{r}(s, t) = \vec{OP} + s\vec{v} + t\vec{w}$ and the implicit equation $ax + by + cz = d$. To get from parametric to implicit, find the normal vector $\vec{n} = \vec{v} \times \vec{w}$. To get from implicit to parametric, find two vectors \vec{v}, \vec{w} normal to the vector \vec{n} . For example, find three points P, Q, R on the surface and form $\vec{u} = \vec{PQ}, \vec{v} = \vec{PR}$.

6.7. The **sphere** $\vec{r}(u, v) = [a, b, c] + [\rho \cos(u) \sin(v), \rho \sin(u) \sin(v), \rho \cos(v)]$ can be brought into the implicit form by finding the center and radius $(x - a)^2 + (y - b)^2 + (z - c)^2 = \rho^2$.

6.8. The parametrization of a graph is $\vec{r}(u, v) = [u, v, f(u, v)]$. It can be written in implicit form as $z - f(x, y) = 0$.

6.9. The surface of revolution is in parametric form given as $\vec{r}(u, v) = [g(v) \cos(u), g(v) \sin(u), v]$. It has the implicit description $\sqrt{x^2 + y^2} = r = g(z)$ which can be rewritten as $x^2 + y^2 = g(z)^2$.

6.10. Here are some level surfaces in cylindrical coordinates:

$r = 1$ is a **cylinder**, $r = |z|$ is a **double cone**, $r^2 = z$ **elliptic paraboloid**, $\theta = 0$ is a **half plane**, $r = \theta$ is a **rolled sheet of paper**.

$r = 2 + \sin(z)$ is an example of a **surface of revolution**.

6.11. Here are some level surfaces described in spherical coordinates:

$\rho = 1$ is a **sphere**, the surface $\phi = \pi/4$ is a **single cone**, $\rho = \phi$ is an **apple shaped surface** and $\rho = 2 + \cos(3\theta) \sin(\phi)$ is an example of a **bumpy sphere**.

HOMEWORK

This homework is due on Tuesday, 7/8/2025

Problem 6.1: Find a parametrization $\vec{r}(t, s)$ for the plane which contains the point $P = (2, 2, 2)$ and the line through $Q = (2, 2, 1)$ and $R = (1, 3, 5)$.

Problem 6.2: Plot the surface with the parametrization

$$\vec{r}(u, v) = [(3 + v \cos(u/2)) \cos(u), (3 + v \cos(u/2)) \sin(u), -v \sin(u/2)]$$

with $-2 \leq v \leq 2$ and $0 \leq u \leq 2\pi$. You can use technology if you like. Do you recognize the shape of the surface?

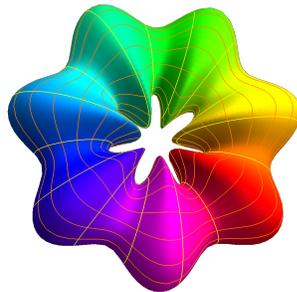
Problem 6.3: Find a parametrizations of the lower half of the ellipsoid $x^2/36 + y^2/16 + (z-4)^2 = 1$ by using that the surface is a graph $z = f(x, y)$ on a suitable domain, then find a second parametrization but use angles ϕ, θ similarly as for the sphere.

Problem 6.4: Find a parametrisation of the **torus candy** (www.math-candy.com) given as the set of points which have distance $6 + 2 \cos(7\theta)$ from the circle

$$[10 \cos(\theta), 10 \sin(\theta), 0],$$

where θ is the angle occurring in cylindrical and spherical coordinates.

Hint: Use r , the distance of a point (x, y, z) to the z -axis. This distance is $r = (10 + (6 + 2 \cos(7\theta)) \cos(\psi))$ if ψ is the angle for the circle winding around the candy. You can also use that $z = (6 + 2 \cos(7\theta)) \sin(\psi)$. To finish the parametrization problem, translate back to Cartesian coordinates.



Problem 6.5: Describe in words or draw a sketch of the surface whose equation is $\rho = |\sin(5\phi)|$ in spherical coordinates (ρ, θ, ϕ) .