

# MULTIVARIABLE CALCULUS

MATH S-21A

## Unit 11: Chain rule

### LECTURE

**11.1.** If  $f$  and  $g$  are functions of  $t$ , the **single variable chain rule** tells us that  $\frac{d}{dt}f(g(t)) = f'(g(t))g'(t)$ . For example,  $\frac{d}{dt}\sin(\log(t)) = \cos(\log(t))/t$ . The rule can be proven by linearization of the functions  $f$  and  $g$  and verifying the chain rule in the linear case. The **chain rule** is also useful:

**11.2.** To find  $\arccos'(x)$  for example, we differentiate  $x = \cos(\arccos(x))$  to get  $1 = d/dx \cos(\arccos(x)) = -\sin(\arccos(x)) \arccos'(x) = -\sqrt{1 - \cos^2(\arccos(x))} \arccos'(x) = -\sqrt{1 - x^2} \arccos'(x)$  so that  $\arccos'(x) = -1/\sqrt{1 - x^2}$ .

**Definition:** Define the **gradient**  $\nabla f(x, y) = [f_x(x, y), f_y(x, y)]$  or  $\nabla f(x, y, z) = [f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)]$ . It is the analog of the derivative in higher dimensions.

**11.3.** If  $\vec{r}(t)$  is curve and  $f$  is a function of several variables we get a function  $t \mapsto f(\vec{r}(t))$  of one variable. Similarly, if  $\vec{r}(t)$  is a parametrization of a planar curve  $f$  is a function of two variables, then  $t \mapsto f(\vec{r}(t))$  is a function of one variable.

**Theorem:**  $\frac{d}{dt}f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$ .

**Proof.** When written out in two dimensions, it is

$$\frac{d}{dt}f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t).$$

The identity

$$\frac{f(x(t+h), y(t+h)) - f(x(t), y(t))}{h} = \frac{f(x(t+h), y(t+h)) - f(x(t), y(t+h))}{h} + \frac{f(x(t), y(t+h)) - f(x(t), y(t))}{h}$$

holds for every  $h > 0$ . The left hand side converges to  $\frac{d}{dt}f(x(t), y(t))$  in the limit  $h \rightarrow 0$  and the right hand side to  $f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)$  using the single variable chain rule twice. Here is the proof of the later, when we differentiate  $f$  with respect to  $t$  and  $y$  is treated as a constant:

$$\frac{f(x(t+h)) - f(x(t))}{h} = \frac{[f(x(t) + (x(t+h) - x(t))) - f(x(t))]}{[x(t+h) - x(t)]} \cdot \frac{[x(t+h) - x(t)]}{h}.$$

Write  $H(t) = x(t+h) - x(t)$  in the first part on the right hand side.

$$\frac{f(x(t+h)) - f(x(t))}{h} = \frac{[f(x(t) + H) - f(x(t))]}{H} \cdot \frac{x(t+h) - x(t)}{h}.$$

As  $h \rightarrow 0$ , we also have  $H \rightarrow 0$  and the first part goes to  $f'(x(t))$  and the second factor to  $x'(t)$ .

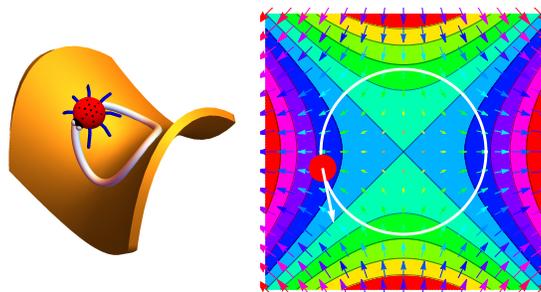
**11.4.** The chain rule is powerful because it implies **all other differentiation rules** like the addition and product rule.  $f(x, y) = x + y, x = u(t), y = v(t), d/dt(x + y) = f_x u' + f_y v' = u' + v'$ .

$f(x, y) = xy, x = u(t), y = v(t), d/dt(xy) = f_x u' + f_y v' = vu' + uv'$ .

**11.5.** We mentioned that the chain rule can be derived from linearization. Let us look at this case: if  $f$  is a linear function  $f(x, y) = ax + by - c$  and if  $\vec{r}(t) = [x_0 + tu, y_0 + tv]$  is a line, then  $\frac{d}{dt}f(\vec{r}(t)) = \frac{d}{dt}(a(x_0 + tu) + b(y_0 + tv)) = au + bv$  and this is the dot product of  $\nabla f = (a, b)$  with  $\vec{r}'(t) = (u, v)$ . Because the chain rule only refers to the derivatives of the functions and the linearization too, the chain rule is also true for general functions.

EXAMPLES

**11.6.** A ladybug moves on a circle  $\vec{r}(t) = [\cos(t), \sin(t)]$  on a table with temperature distribution  $f(x, y) = x^2 - y^3$ . Find the rate of change of the temperature  $\nabla f(x, y) = (2x, -3y^2), \vec{r}'(t) = (-\sin(t), \cos(t))$   $d/dt f(\vec{r}(t)) = \nabla T(\vec{r}(t)) \cdot \vec{r}'(t) = (2 \cos(t), -3 \sin(t)^2) \cdot (-\sin(t), \cos(t)) = -2 \cos(t) \sin(t) - 3 \sin^2(t) \cos(t)$ .



**11.7.** From  $f(x, y) = 0$ , one can express  $y$  as a function of  $x$ , at least near a point where  $f_y$  is not zero. From  $\frac{d}{dx}f(x, y(x)) = \nabla f \cdot (1, y'(x)) = f_x + f_y y' = 0$ , we obtain  $y' = -f_x/f_y$ . Even so, we do not know  $y(x)$ , we can compute its derivative! Implicit differentiation works also in three variables. The equation  $f(x, y, z) = c$  defines a surface. Near a point where  $f_z$  is not zero, the surface can be described as a graph  $z = z(x, y)$ . We can compute the derivative  $z_x$  without actually knowing the function  $z(x, y)$ . To do so, we consider  $y$  a fixed parameter and compute, using the chain rule

$$f_x(x, y, z(x, y)) \cdot 1 + f_y(x, y, z(x, y)) \cdot 0 + f_z(x, y, z(x, y)) \cdot z_x(x, y) = 0$$

so that  $z_x(x, y) = -f_x(x, y, z)/f_z(x, y, z)$ . This works at points where  $f_z$  is not zero.

**11.8.** The surface  $f(x, y, z) = x^2 + y^2/4 + z^2/9 = 6$  is an ellipsoid. Compute  $z_x(x, y)$  at the point  $(x, y, z) = (2, 1, 1)$ .

**Solution:**  $z_x(x, y) = -f_x(2, 1, 1)/f_z(2, 1, 1) = -4/(2/9) = -18$ .

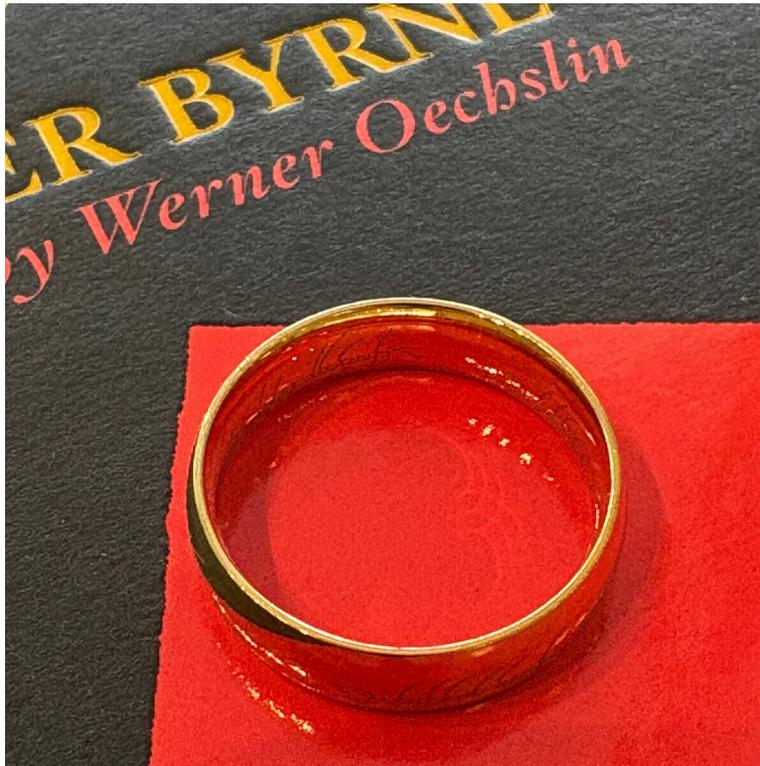
## HOMEWORK

This homework is due on Tuesday, 7/15/2025 in class.

**Problem 11.1:** You know that  $d/dt f(\vec{r}(t)) = 19$  at  $t = 7$  if  $\vec{r}(t) = [t, t]$  and  $d/dt f(\vec{r}(t)) = 11$  at  $t = 7$ .  $\vec{r}(t) = [t, 14 - t]$ . Find the gradient of  $f$  at  $(7, 7)$ .

**Problem 11.2:** a) Define the function  $f(x, y) = x/y$ . Compute the gradient of  $f$  and tell what the chain rule  $\frac{d}{dt} f(x(t), y(t))$  tells. It is a rule you know from calculus. (The chain rule is the ring of power which binds them all.)

b) The pressure in the space at the position  $(x, y, z)$  is  $p(x, y, z) = x^2 + y^2 - z^3$  and the trajectory of an observer is the curve  $\vec{r}(t) = [t, t, 1/t]$ . Using the chain rule, compute the rate of change of the pressure the observer measures at time  $t = 2$ .



Oliver carries the ring of power. One can see that it is real by noticing that it shows three caustics!

**Problem 11.3:** The chain rule is closely related to linearization. Lets get back to linearization a bit: A farm costs  $f(x, y)$ , where  $x$  is the number of cows and  $y$  is the number of ducks. There are 10 cows and 20 ducks and  $f(10, 20) = 1000000$ . We know that  $f_x(x, y) = 2x$  and  $f_y(x, y) = y^2$  for all  $x, y$ . Estimate  $f(12, 19)$ .

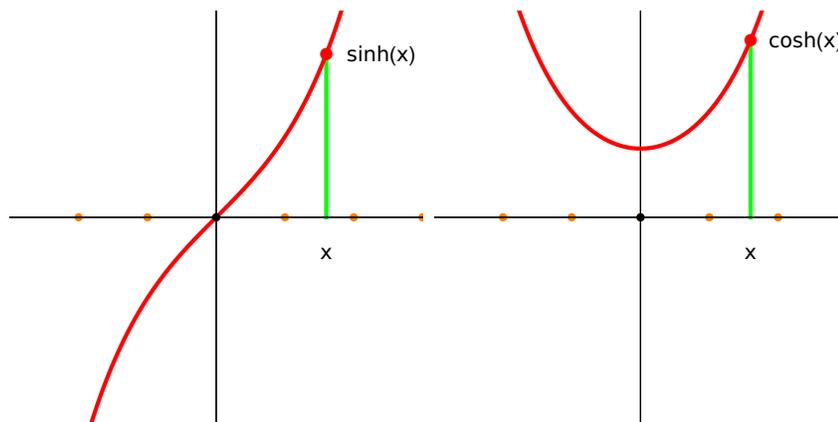
Here is a song out of this:

*”Old MacDonald had a million dollar farm, E-I-E-I-O,  
and on that farm he had  $x = 10$  cows, E-I-E-I-O,  
and on that farm he had  $y = 20$  ducks, E-I-E-I-O,  
with  $f_x = 2x$  here and  $f_y = y^2$  there,  
and here two cows more, and there a duck less,  
how much does the farm cost now, E-I-E-I-O?”*

**Problem 11.4:** Find, using implicit differentiation the derivative  $d/dx \operatorname{arctanh}(x)$ , where

$$\tanh(x) = \sinh(x) / \cosh(x) .$$

The **hyperbolic sine** and **hyperbolic cosine** are defined as  $\sinh(x) = (e^x - e^{-x})/2$  and  $\cosh(x) = (e^x + e^{-x})/2$ . We have  $\sinh' = \cosh$  and  $\cosh' = \sinh$  and  $\cosh^2(x) - \sinh^2(x) = 1$ .



**Problem 11.5:** The equation  $f(x, y, z) = e^{xyz} + z = 1 + e$  implicitly defines  $z$  as a function  $z = g(x, y)$  of  $x$  and  $y$ . Find formulas (in terms of  $x, y$  and  $z$ ) for  $g_x(x, y)$  and  $g_y(x, y)$ . Estimate  $g(1.01, 0.999)$  using linear approximation.