# Unit 13: Extrema

LECTURE

**13.1.** In applications, we need to **maximize** or **minimize** a function $f$. As in single variable calculus, we look for points, where the **derivative** is zero. We use the **Fermat principle** telling that at points, where the derivative is non-zero we can increase the function by going into the direction of the gradient. As in one dimensions, we can then use the **second derivative test** to classify the extrema, like **local maximum** at $-1$ and the **local minimum** at 1.

> **Definition:** A point $(a, b)$ in the plane is called a **critical point** of a function $f(x, y)$ if $\nabla f(a, b) = [0, 0]$.

**13.2.** The **Fermat principle** in two dimensions tells:

> If $\nabla f(x, y)$ is not zero, then $(x, y)$ is not a maximum or minimum.

**13.3. Proof.** Take the directional derivative in the direction $\vec{v} = \nabla f / |\nabla f|$ at $\vec{x} = (x, y)$. Then $D_{\vec{v}} f = \nabla f \cdot \vec{v} = |\nabla f| > 0$. This means that $f(\vec{x} + \epsilon \vec{v}) > f(\vec{x})$ and $f(\vec{x} - \epsilon \vec{v}) < f(\vec{x})$ for small $\epsilon$ and $\vec{x}$ is neither a maximum nor a minimum. QED

**13.4.** Points, where $f$ or its derivative is not defined are not included in the list of critical points. These points are treated separately. For $f(x, y) = |x| + |y|$ for example, we would disregard at first two coordinate axis where the function is not differentiable and study $f$ on these lines separately. If nothing is said otherwise, we assume that $f$ is smooth, meaning that it can be differentiated arbitrarily many times. Points, where the function has no derivatives are not considered to be part of the domain and are studied separately. For the continuous function $f(x, y) = 1/\log(|xy|)$ for example, we would study $x = 0$, or $y = 0$ or $xy = 1$ separately.

**13.5.** In one dimension, you have used the condition $f'(x) = 0$, $f''(x) > 0$ to get a local minimum and $f'(x) = 0$, $f''(x) < 0$ to assure a local maximum. If $f'(x) = 0$, $f''(x) = 0$, the nature of the critical point is undetermined. It could be a maximum like for $f(x) = -x^4$, a minimum like for $f(x) = x^4$ or a flat **inflection point**, like for $f(x) = x^3$, where we have neither a maximum nor a minimum.

**Definition:** If $f(x, y)$ is a function of two variables with a critical point $(a, b)$, the value $D = f_{xx}f_{yy} - f_{xy}^2$ is called the **discriminant** of the critical point.

**13.6.** The discriminant can be remembered well if seen as the determinant of the **Hessian matrix** $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ which combines all the second partial derivatives in one entity, a matrix. [1] As of default, we always assume that functions are twice continuously differentiable. Here is the **second derivative test**:

**Theorem:** Assume $(a, b)$ is a critical point for $f(x, y)$.
If $D > 0$ and $f_{xx}(a, b) > 0$ then $(a, b)$ is a local min.
If $D > 0$ and $f_{xx}(a, b) < 0$ then $(a, b)$ is a local max.
If $D < 0$ then $(a, b)$ is a saddle point.

**13.7.** If $D \neq 0$ at all critical points, the function $f$ is called **Morse**. The Morse condition is nice as for $D = 0$, we need to look at higher derivatives or other insight to be able to determine the nature of the critical point.

**13.8.** To find the max or min of $f(x, y)$ on a domain, determine all critical points **in the interior the domain**, and compare their values with maxima or minima **at the boundary**. We will see in the next unit how to get extrema on the boundary.

**13.9.** Sometimes, we want to find the overall maximum and not only the local ones.

**Definition:** A point $(a, b)$ in the plane is called a **global maximum** of $f(x, y)$ if $f(x, y) \leq f(a, b)$ for all $(x, y)$. For example, the point $(0, 0)$ is a global maximum of the function $f(x, y) = 1 - x^2 - y^2$. Similarly, we call $(a, b)$ a **global minimum**, if $f(x, y) \geq f(a, b)$ for all $(x, y)$.

[2]

## EXAMPLES

**13.10.** Find the critical points of $f(x, y) = x^4 + y^4 - 4xy + 2$. The gradient is $\nabla f(x, y) = [4(x^3 - y), 4(y^3 - x)]$. This is $[0, 0]$ at the points $(0, 0), (1, 1), (-1, -1)$. These are the critical points.

**13.11.** $f(x, y) = \sin(x^2 + y) + y$. The gradient is $\nabla f(x, y) = [2x\cos(x^2 + y), \cos(x^2 + y) + 1]$. For a critical points, we must have $x = 0$ and $\cos(y) + 1 = 0$ which means $\pi + k2\pi$. The critical points are at $\ldots (0, -\pi), (0, \pi), (0, 3\pi), \ldots$. There are infinitely many critical points here.

**13.12.** The graph of $f(x, y) = (x^2 + y^2)e^{-x^2-y^2}$ looks like a volcano. The gradient $\nabla f = [2x - 2x(x^2 + y^2), 2y - 2y(x^2 + y^2)]e^{-x^2-y^2}$ vanishes at $(0, 0)$ and on the circle $x^2 + y^2 = 1$. This function has a continuum of critical points.

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[1] Matrix comes from Mater = womb of a mother as matrices are the mother of determinants!
[2] We avoid "absolute maximum" as this would suggest to look for the maximum of $|f|$. Compare for example when looking at absolute convergence of series.

**13.13.** The function $f(x, y) = y^2/2 - g\cos(x)$ is the energy of the pendulum. The variable $g$ is a constant and related to the gravitational strength. We have $\nabla f = (y, -g\sin(x)) = [(0, 0]$ for

$$(x, y) = \ldots, (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0), \ldots .$$

These points are equilibrium points, the angles for which the pendulum is at rest.

**13.14.** The function $f(x, y) = |x| + |y|$ is smooth on the first quadrant $\{x > 0, y > 0\}$ where it reduces to $f(x, y) = x + y$. The function does not have critical points away from the coordinate axes. Any local maximum or local minimum must be contained on a coordinate axes. On the $x$-axes we have $f(x, 0) = |x|$ which is a function that only smooth if $x \neq 0$ and which has a minimum at $x = 0$. Similarly for $f(0, y) = |y|$ we see that only $y = 0$ is a candidate for a minimum or maximum. We see that the function has a minimum $(0, 0)$. it is not in the domain of $f$, where both $f$ and $\nabla f$ are defined. The minimum needed to be detected by a more elaborate analysis. This happens often in linear programming problems. For us in calculus, we most of the time assume that functions are smooth and nice. The function $(x^n + n^{-n})^{(1/n)}$ for example is smooth and close to the function $f(x) = |x|$

**13.15.** The function $f(x, y) = x^3/3 - x - (y^3/3 - y)$ has a graph which looks like a "napkin". It has the gradient $\nabla f(x, y) = [x^2 - 1, -y^2 + 1]$. There are 4 critical points $(1, 1), (-1, 1), (1, -1)$ and $(-1, -1)$. The Hessian matrix which includes all partial derivatives is $H = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$.

For $(1, 1)$ we have $D = -4$ and so a saddle point,
For $(-1, 1)$ we have $D = 4$, $f_{xx} = -2$ and so a local maximum,
For $(1, -1)$ we have $D = 4$, $f_{xx} = 2$ and so a local minimum.
For $(-1, -1)$ we have $D = -4$ and so a saddle point. The function has a local maximum, a local minimum as well as 2 saddle points.
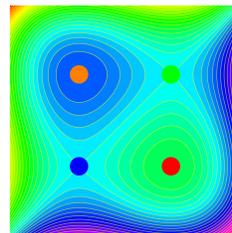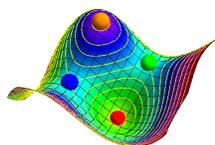


**13.16.** Find the maximum of $f(x, y) = 2x^2 - x^3 - y^2$ on $y \geq -1$. With $\nabla f(x, y) = 4x - 3x^2, -2y)$, the critical points are $(4/3, 0)$ and $(0, 0)$. The Hessian is $H(x, y) = \begin{bmatrix} 4 - 6x & 0 \\ 0 & -2 \end{bmatrix}$. At $(0, 0)$, the discriminant is $-8$ so that this is a saddle point. At $(4/3, 0)$, the discriminant is $8$ and $H_{11} = 4/3$, so that $(4/3, 0)$ is a local maximum. We have now also to look at the boundary $y = -1$ where the function is $g(x) = f(x, -1) = 2x^2 - x^3 - 1$. Since $g'(x) = 0$ at $x = 0, 4/3$, where $0$ is a local minimum, and $4/3$ is a local maximum on the line $y = -1$. Comparing $f(4/3, 0), f(4/3, -1)$ shows that $(4/3, 0)$ is the global maximum.

**13.17.** Find the global maxima and minima of $f(x,y) = x^4 + y^4 - 2x^2 - 2y^2$ **Solution:** the function has no global maximum. This can be seen by restricting the function to the $x$-axis, where $f(x,0) = x^4 - 2x^2$ is a function without maximum. The function has four global minima however. They are located on the 4 points $(\pm 1, \pm 1)$. The best way to see this is to note that $f(x,y) = (x^2 - 1)^2 + (y - 1)^2 - 2$ which is minimal when $x^2 = 1, y^2 = 1$.

# Homework

This homework is due on Tuesday, 7/22/2025.

**Problem 13.1:** Find all the extrema of the function
$$f(x,y) = 6x * y + 2x^2 * y + 2x * y^2$$
they are maxima, minima or saddle points.

**Problem 13.2:** Consider the parametrized surface $\vec{r}(u,v) = [1 + u^3, v^2, uv]$. Where is the temperature $T(x,y,z) = 7 + x + 12y - 24z$ minimal on this surface? To find the minimum, minimize the function $f(u,v) = T(\vec{r}(u,v))$. Find all local maxima, local minima or saddle points of $f$.

**Problem 13.3:** Do this problem as it is an old classic problem and it will be continued in the next unit Find and classify all the extrema of the function
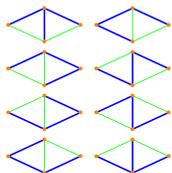$f(x,y) = e^{-x^2 - y^2}(x^2 + 2y^2)$.

**Problem 13.4:** Find all extrema of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 13$ and characterize them. Do you find a global maximum or global minimum among them?

**Problem 13.5:** Graph theorists are fond of at the **Tutte polynomial** $f(x,y)$ of a network. We work with the Tutte polynomial
$$f(x,y) = x + 2x^2 + x^3 + y + 2xy + y^2$$
of the **Kite network**. Classify using the second derivative test.

**Remark.** The polynomial is useful: $xf(1-x,0)$ tells in how many ways one can color the nodes of the network with $x$ colors and $f(1,1)$ tells how many spanning trees there are. This picture illustrates that the number of spanning trees of the kite graph is $f(1,1) = 8$ as you see the 8 possible trees.