MULTIVARIABLE CALCULUS

MATH S-21A

Unit 17: Triple integrals

Lecture

17.1. A **solid** is a three dimensional region. Examples are **solid balls** like $E = B_\rho = \{x^2 + y^2 + z^2 \leq \rho^2\}$ or the **unit cube** $E = \{0 \leq |x| \leq 1, 0 \leq |y| \leq 1, 0 \leq |z| \leq 1\}$. In general, we assume that a solid is bound by piecewise smooth surfaces. Sets like the **Mandelbulb** in space would be more tricky to deal with. A solid is **bounded** if it is contained in some solid ball B_ρ of radius $\rho > 0$.

Definition: If $f(x, y, z)$ is continuous and E is a **bounded solid** in $\mathbb{R}^3$, then $\iiint_E f(x, y, z)\ dxdydz$ is defined as the $n \to \infty$ limit of the **Riemann sum**

$$\sum_{(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}) \in E} f(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}) \frac{1}{n^3} .$$

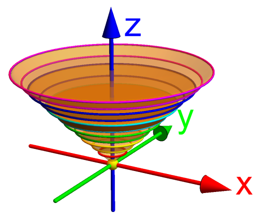
Triple integrals can be evaluated by iterated single integrals. Here is an example:

17.2. If E is the box $\{x \in [0, 1], y \in [0, 1], z \in [0, 1]\}$ and $f(x, y, z) = 24x^2y^3z$.

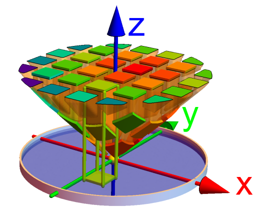
$$\int_0^1 \int_0^1 \int_0^1 24x^2y^3z\ dz\ dy\ dx .$$

To evaluate the integral, start from the inside $\int_0^1 24x^2y^3z\ dz = 12x^2y^3$, then then integrate the middle layer, $\int_0^1 12x^2y^3\ dy = 3x^2$ and finally and finally handle the most outer layer: $\int_0^1 3x^2 dx$ =1.
For the inner integral, $x = x_0$ and $y = y_0$ are fixed. The middle integral now computes the contribution over a slice $z = z_0$ intersected with R. The outer integral sums up all these slice contributions.

17.3. Triple integrals can be computed by reducing to a single integral or two a double integral:



The **burger method** slices the solid a line and computes $\int_a^b \iint_{R(z)} f(x,y,z)\ dA\ dz$, where $g(z)$ is a double integral giving the values when integrating over cheese, meet or tomato. The **fries method** eats up fries going from $g(x,y)$ to $h(x,y)$ over a region R. We have $\iint_R [\int_{g(x,y)}^{h(x,y)} f(x,y,z)\ dz]\ dA$.



17.4. A special case is the **signed volume**

$$\iint_R \int_0^{f(x,y)} 1\ dzdxdy$$

below the graph of a function $f(x,y)$ and above a region R in the xy-plane. The triple integral reduces to the double integral $\iint_R f(x,y)\ dA$. Triple integral give us more flexibility: we can replace the constant density function 1 with a function $f(x,y,z)$. If $f(x,y,z)$ is interpreted as a **mass density** at the point (x,y,z), then the integral would be the **mass** of the solid. It can also be negative like when we deal with charge density.

17.5. The problem of computing volumes has been been worked on by **Archimedes (287-212 BC)**. His **method of exhaustion** was a precursor of Riemann sums. It allowed him to find areas, volumes and surface areas in many cases without calculus. One idea is **comparison**. Already the **Archimedes principle** relating volume to the amount of displayed water is such an idea. The **displacement method** is a **comparison technique**: the area of a sphere is the area of the cylinder enclosing it. The volume of a sphere is the volume of the complement of a cone in that cylinder. **Cavalieri (1598-1647)** would build on Archimedes ideas and determine area and volume using tricks now called the **Cavalieri principle**. An example already due to Archimedes is the computation of the volume the half sphere of radius R, cut away a cone of height and radius R from a cylinder of height R and radius R. At height z, this body has a cross section with area $R^2\pi - r^2\pi$. If we cut the half sphere at height z, we obtain a disc of area $(R^2 - r^2)\pi$. Because these areas are the same, the volume of the half-sphere is the same as the cylinder minus the cone: $\pi R^3 - \pi R^3/3 = 2\pi R^3/3$ and the volume of the sphere is $4\pi R^3/3$. **Newton (1643-1727) and Leibniz(1646-1716)** developed calculus independently and so provided a new **analytic tool** which made it possible to compute integrals through "anti-derivation". Suddenly, it became possible to find integrals using analytic tools, which even would escape the ingenuity of Archimedes. We can do this also in higher dimensions.

Examples

17.6. Find the volume of the unit sphere. **Solution:** The sphere is sandwiched between the graphs of two functions obtained by solving for z. Let R be the unit disc in the xy plane. If we use the **sandwich method**, we get

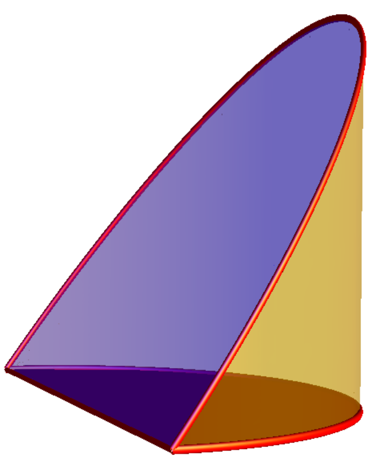
$$V = \int\int_R [\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1 dz] dA \; .$$

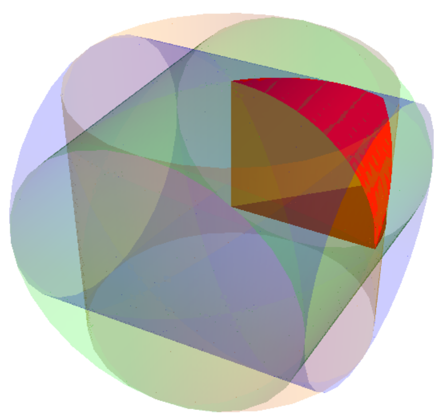
which gives a double integral $\int\int_R 2\sqrt{1-x^2-y^2} \; dA$ which is of course best solved in polar coordinates. We have $\int_0^{2\pi} \int_0^1 \sqrt{1-r^2} r \; dr d\theta = 4\pi/3$.
With the **washer method** which is in this case also called **disc method**, we slice along the z axes and get a disc of radius $\sqrt{1-z^2}$ with area $\pi(1-z^2)$. This is a method suitable for single variable calculus because we get directly $\int_{-1}^1 \pi(1-z^2) \; dz = 4\pi/3$.

17.7. The mass of a body with mass density $\rho(x,y,z)$ is defined as $\int\int\int_R \rho(x,y,z) \; dV$. For bodies with constant density ρ, the mass is ρV, where V is the volume. Compute the mass of a body which is bounded by the parabolic cylinder $z = 4 - x^2$, and the planes $x = 0, y = 0, y = 6, z = 0$ if the density of the body is z. **Solution:**

$$\begin{aligned} \int_0^2 \int_0^6 \int_0^{4-x^2} z \; dz \; dy \; dx &= \int_0^2 \int_0^6 (4-x^2)^2/2 \; dy dx \\ = \; 6 \int_0^2 (4-x^2)^2/2 \; dx &= 6(\frac{x^5}{5} - \frac{8x^3}{3} + 16x)|_0^2 = 2 \cdot 512/5 \end{aligned}$$

17.8. The solid region bound by $x^2 + y^2 = 1$, $x = z$ and $z = 0$ is called the **hoof of Archimedes**. It is historically significant because it is one of the first examples, on which Archimedes probed a Riemann sum integration technique. It appears in every calculus text book. Find the volume of the hoof. **Solution**. Look from the situation from above and picture it in the xy-plane. You see a half disc R. It is the floor of the solid. The roof is the function $z = x$. We have to integrate $\int\int_R x \; dxdy$. We got a double integral problems which is best done in polar coordinates; $\int_{-\pi/2}^{\pi/2} \int_0^1 r^2 \cos(\theta) \; dr d\theta = 2/3$.





17.9. Finding the volume of the solid region bound by the three cylinders $x^2 + y^2 = 1$, $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ is one of the most famous volume integration problems going back to Archimedes.
Solution: look at 1/16'th of the body given in cylindrical coordinates $0 \leq \theta \leq \pi/4, r \leq$

$1, z > 0$. The roof is $z = \sqrt{1-x^2}$ because above the "one eighth disc" R only the cylinder $x^2 + z^2 = 1$ matters. The polar integration problem

$$16 \int_0^{\pi/4} \int_0^1 \sqrt{1 - r^2\cos^2(\theta)}r \; drd\theta$$

has an inner r-integral of $(16/3)(1 - \sin(\theta)^3)/\cos^2(\theta)$. Integrating this over θ can be done by integrating $(1 + \sin(x)^3)\sec^2(x)$ by parts using $\tan'(x) = \sec^2(x)$ leading to the anti derivative $\cos(x) + \sec(x) + \tan(x)$. The result is $16 - 8\sqrt{2}$.

Homework

This homework is due on Tuesday, 7/29/2025.

Problem 17.1: Evaluate the triple integral

$$\int_0^4 \int_0^z \int_0^{4y} 6z^3 \; dxdydz \; .$$

Problem 17.2: What is $\int_0^1 \int_0^1 \int_y^1 4xe^{-z^2} \, dzdydx$?

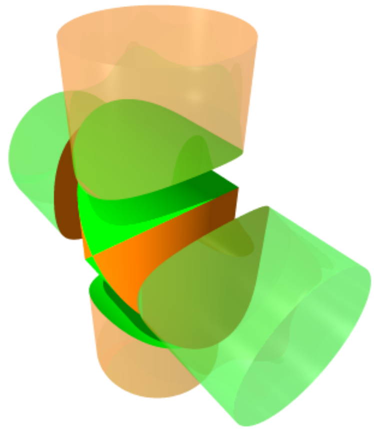
Problem 17.3: Find the **moment of inertia** $\iint_E (y^2 + z^2) \; dV$ of a cone

$$E = \{y^2 + z^2 \leq x^2 \; 0 \leq x \leq 15 \} \; ,$$

which has the x-axis as its center of symmetry.

Problem 17.4: Integrate $f(x, y, z) = x^2 + y^2 - z$ over the tetrahedron with vertices
$(0, 0, 0), (4, 4, 0), (0, 4, 0), (0, 0, 12)$.

Problem 17.5: This is a classic problem of Archimedes: what is the volume of the body obtained by intersecting the solid cylinders $x^2 + z^2 \leq 9$ and $y^2 + z^2 \leq 9$?



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