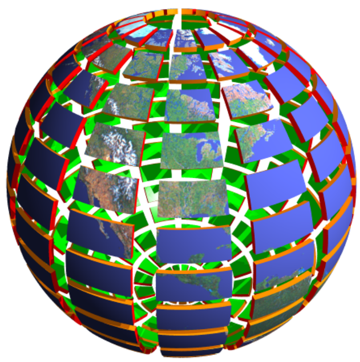
Unit 18: Spherical integrals

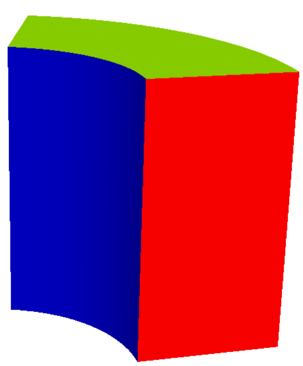
Lecture

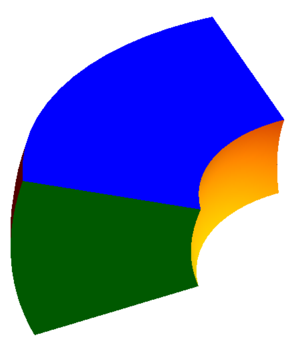
18.1. If a solid has some symmetry, cylindrical and spherical coordinate systems can help to integrate.



Definition: Cylindrical coordinates are coordinates in $\mathbb{R}^3$, where **polar coordinates** are used in the xy-plane while the z-coordinate is not changed. The coordinate transformation $T(r,\theta,z) = (r\cos(\theta), r\sin(\theta), z)$, produces the integration factor $\boxed{r}$. It is the same factor than in polar coordinates.

$$\iiint_{T(R)} f(x,y,z)\, dxdydz = \iiint_R g(r,\theta,z)\, \boxed{\text{r}}\, drd\theta dz$$





Definition: Spherical coordinates use the radius $\rho \geq 0$, the distance to the origin as well as two **Euler angles**: $0 \leq \theta < 2\pi$ the polar angle and $0 \leq \phi \leq \pi$, the angle between the vector and the positive z axis. The coordinate change is

$$T : (x, y, z) = (\rho \cos(\theta) \sin(\phi), \rho \sin(\theta) \sin(\phi), \rho \cos(\phi)) .$$

The integration factor measures the volume of a **spherical wedge** which is $d\rho \cdot \rho \sin(\phi) \cdot d\theta \cdot \rho d\phi = \rho^2 \sin(\phi) d\theta d\phi d\rho$.

$$\iiint_{T(R)} f(x, y, z) \, dxdydz = \iiint_R g(\rho, \theta, z) \boxed{\rho^2 \sin(\phi)} \, d\rho d\theta d\phi$$

A ball of radius R has the volume

$$\int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin(\phi) \, d\phi d\theta d\rho .$$

The most inner integral $\int_0^{\pi} \rho^2 \sin(\phi) d\phi = -\rho^2 \cos(\phi)|_0^{\pi} = 2\rho^2$. The next layer is, because ϕ does not appear: $\int_0^{2\pi} 2\rho^2 \, d\phi = 4\pi\rho^2$. The final integral is $\int_0^R 4\pi\rho^2 \, d\rho = 4\pi R^3/3$.

Definition: The moment of inertia of a body G with respect to an axis L is defined as the triple integral $\int\int\int_G r(x, y, z)^2 \, dzdydx$, where $r(x, y, z) = \rho \sin(\phi)$ is the distance from the axis L.

Examples

18.2. For a ball of radius R we obtain with respect to the z-axis:

$$I = \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin^2(\phi) \rho^2 \sin(\phi) \, d\phi d\theta d\rho$$

$$= (\int_0^{\pi} \sin^3(\phi) \, d\phi)(\int_0^R \rho^4 \, dr)(\int_0^{2\pi} \, d\theta)$$

$$= (\int_0^{\pi} \sin(\phi)(1 - \cos^2(\phi)) \, d\phi)(\int_0^R \rho^4 \, dr)(\int_0^{2pi} \, d\theta)$$

$$= (-\cos(\phi) + \cos(\phi)^3/3)|_0^{\pi}(L^5/5)(2\pi) = \frac{4}{3} \cdot \frac{R^5}{5} \cdot 2\pi = \frac{8\pi R^5}{15} .$$

18.3. If the sphere rotates with angular velocity ω, then $I\omega^2/2$ is the **kinetic energy** of that sphere. The moment of inertia of our **earth** for example is $8 \cdot 10^{37} kgm^2$. The angular velocity is $\omega = 2\pi/day = 2\pi/(86400s)$. The rotational energy is $8 \cdot 10^{37} kgm^2/(7464960000s^2) \sim 10^{29} J \sim 2.5 \cdot 10^{24} kcal$.

18.4. Find the volume and the center of mass of a diamond, the intersection of the unit sphere with the cone given in cylindrical coordinates as $z = \sqrt{3}r$.
Solution: we use spherical coordinates to find the center of mass

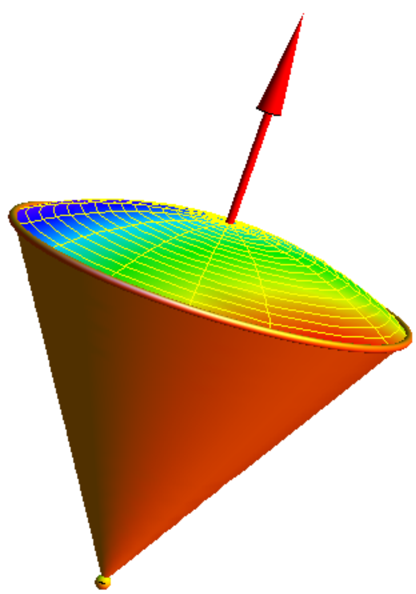
$$\begin{aligned}
\overline{x} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \sin^2(\phi)\cos(\theta)\ d\phi d\theta d\rho \frac{1}{V} = 0 \\
\overline{y} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \sin^2(\phi)\sin(\theta)\ d\phi d\theta d\rho \frac{1}{V} = 0 \\
\overline{z} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \cos(\phi)\sin(\phi)\ d\phi d\theta d\rho \frac{1}{V} = \frac{2\pi}{32V}
\end{aligned}$$

18.5. Find $\int\int\int_R z^2\ dV$ for the solid obtained by intersecting $\{1 \leq x^2 + y^2 + z^2 \leq 4\}$ with the double cone $\{z^2 \geq x^2 + y^2\}$.
Solution: since the result for the double cone is twice the result for the single cone, we work with the diamond shaped region R in $\{z > 0\}$ and multiply the result at the end with 2. In spherical coordinates, the solid R is given by $1 \leq \rho \leq 2$ and $0 \leq \phi \leq \pi/4$. With $z = \rho\cos(\phi)$, we have

$$\int_1^2 \int_0^{2\pi} \int_0^{\pi/4} \rho^4 \cos^2(\phi)\sin(\phi)\ d\phi d\theta d\rho$$

$$= (\frac{2^5}{5} - \frac{1^5}{5}) 2\pi (\frac{-\cos^3(\phi))}{3}|_0^{\pi/4} = 2\pi \frac{31}{5}(1 - 2^{-3/2})\ .$$

The result for the double cone is $\boxed{4\pi(31/5)(1 - 1/\sqrt{2}^3)}$.



Homework

This homework is due on Tuesday, 7/29/2025.

Problem 18.1: A hot air balloon E has the shape $x^2+y^2+z^2 \leq 1, z \geq -1/\sqrt{2}$. The density of the gas is $f(x,y,z) = 1+z$. The pilot computes the amount of $\iiint_E f(x,h,z)\ dV$ using cylindrical coordinates and gets $\int_0^{2\pi} \int_{-1/\sqrt{2}}^1 \int_0^{\sqrt{1-z^2}} (1+z) r dr dz d\theta$. Then he computes the same volume using spherical coordinates $\int_0^{2\pi} \int_0^{3\pi/4} \int_0^1 \rho^2 \sin(\phi)(1+\rho\cos(\phi))\ d\rho d\phi d\theta$. Compute both integrals. You will see that they do not agree. Which of the two integrals correctly computes the volume? What went wrong with the other?

Problem 18.2: Assume the mass density of a solid $E = x^2+y^2-z^2 < 1, -1 < z < 1$ is given by the 8's power of the distance to the z-axes: $\sigma(x,y,z) = r^8 = (x^2+y^2)^4$. Find its mass

$$M = \int \int \int_E (x^2+y^2)^4\ dxdydz\ .$$

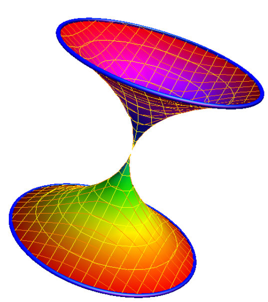
Problem 18.3: A solid is described in spherical coordinates by the inequality $\rho \leq 2\sin(\phi)$. Find its volume.

Problem 18.4: Integrate the function

$$f(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}}$$

over the solid which lies between the spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$, which is in the first octant and which is above the cone $x^2+y^2 = z^2$.

Problem 18.5: Find the volume of the solid $x^2+y^2 \leq z^4, z^2 \leq 4$.



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