MULTIVARIABLE CALCULUS

MATH S-21A

Unit 19: Vector fields

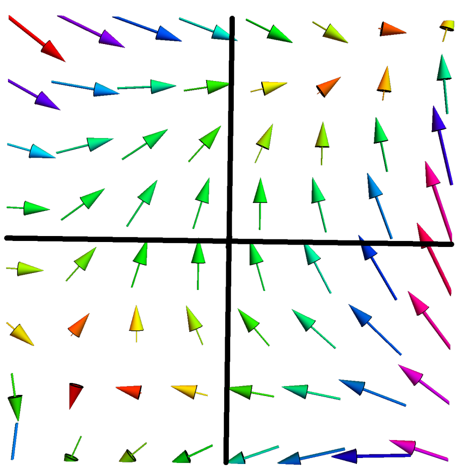
Lecture

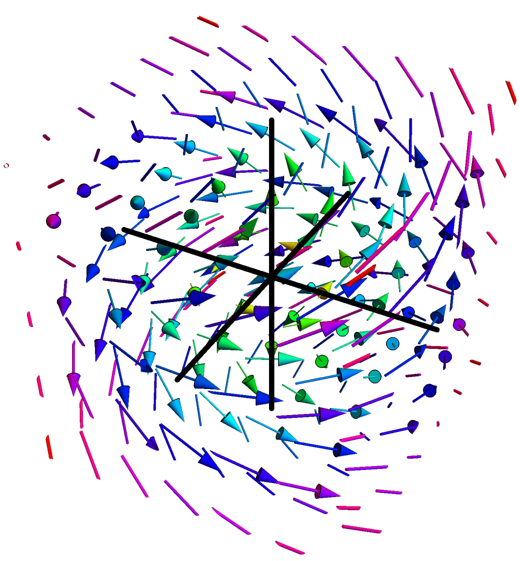
19.1. Real valued functions $f(x,y)$ or $f(x,y,z)$ are also known as **scalar fields**. **Vector valued functions** appear in **curves** $\vec{r}(t)$ and **surfaces** $\vec{r}(u,v)$. A vector-valued function $\vec{F}$ with the same number of components than variables can be seen as a **vector field**. Lets make this more precise:

Definition: A **planar vector field** is a vector-valued map $\vec{F}$ which assigns to every point $(x,y) \in \mathbb{R}^2$ a vector $\vec{F}(x,y) = [P(x,y), Q(x,y)]$. A **vector field in space** is a map, which assigns to each point $(x,y,z) \in \mathbb{R}^3$ a vector $\vec{F}(x,y,z) = [P(x,y,z), Q(x,y,z), R(x,y,z)]$.

19.2. Here are examples of vector fields in two and three dimensions

$$\vec{F}(x,y) = \begin{bmatrix} y - \sin(x) \\ x^3 + \cos(2y) \end{bmatrix}, \vec{F}(x,y,z) = \begin{bmatrix} -y \\ x \\ \sin(z) \end{bmatrix}.$$





Definition: If $f(x,y)$ is a function of two variables, then $\vec{F}(x,y) = \nabla f(x,y)$ is a vector field called the **gradient field** of f. Gradient fields in space are of the form $\vec{F}(x,y,z) = \nabla f(x,y,z)$. They are important!

19.3. When is a vector field a gradient field? Note that Clairaut assured that $\vec{F}(x,y) = [P(x,y), Q(x,y)] = \nabla f(x,y)$ implies $Q_x(x,y) = P_y(x,y)$. If this does not hold at some point, $\vec{F}$ can not be a gradient field.

Clairaut test: If $Q_x(x,y) - P_y(x,y)$ is not zero at some point, then $\vec{F}(x,y) = [P(x,y), Q(x,y)]$ is not a gradient field.

19.4. We will see next week that $\text{curl}(\vec{F}) = Q_x - P_y = 0$ is also sufficient for $\vec{F}$ to be a gradient field if $\vec{F}$ is defined everywhere. How do we get f the function with $\vec{F} = \nabla f$? We will look at examples in class.

Examples

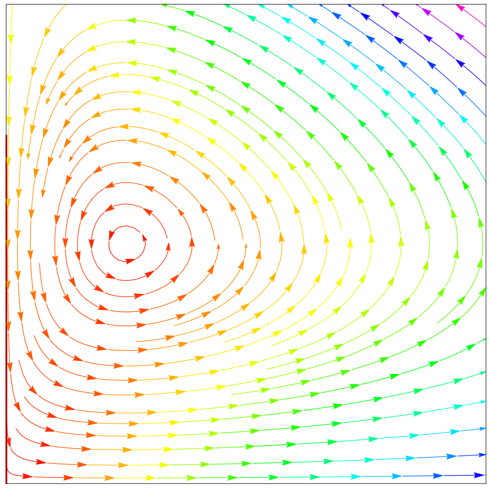
19.5. Is the vector field $\vec{F}(x,y) = [P,Q] = [3x^2y + y + 2, x^3 + x - 1]$ a gradient field? **Solution**: the Clairaut test shows $Q_x - P_y = 0$. We integrate the equation $f_x = P = 3x^2y + y + 2$ and get $f(x,y) = 2x + xy + x^3y + c(y)$. Now take the derivative of this with respect to y to get $x + x^3 + c'(y)$ and compare with $x^3 + x - 1$. We see $c'(y) = -1$ and so $c(y) = -y + c$. We see the solution $\boxed{x^3y + xy - y + 2x}$.

19.6. Is the vector field $\vec{F}(x,y) = [xy, 2xy^2]$ a gradient field? **Solution:** No: $Q_x - P_y = 2y^2 - x$ is not zero.
Vector fields appear naturally when studying differential equations. Here is an example in population dynamics:

19.7. If $x(t)$ is the population of a "prey species" like shrimp and $y(t)$ is the population size of a "predator" like sharks. We have $x'(t) = ax(t) - bx(t)y(t)$ with positive a, b because both more predators and more prey species will lead to prey consumption. The rate of change of $y(t)$ is $y'(t) = -cy(t) + dxy$, where c, d are positive. This can be written using a vector field $\vec{r}' = \vec{F}(\vec{r}(t))$. We have a negative sign in the first part because predators would die out without food. The second term is explained because both more predators as well as more prey leads to a growth of predators through reproduction. A concrete example is the **Volterra-Lodka system**

$$\begin{aligned} \dot{x} &= 0.4x - 0.4xy \\ \dot{y} &= -0.1y + 0.2xy \, , \end{aligned}$$

where $\vec{F}(x,y) = [0.4x - 0.4xy, -0.1y + 0.2xy]$. Volterra explained with such systems the oscillation of fish populations in the Mediterranean sea. At any specific point $\vec{r}(x,y) = [x(t), y(t)]$, there is a curve $= \vec{r}(t) = [x(t), y(t)]$ through that point for which the tangent $\vec{r}\,'(t) = (x'(t), y'(t)$ is the vector field.

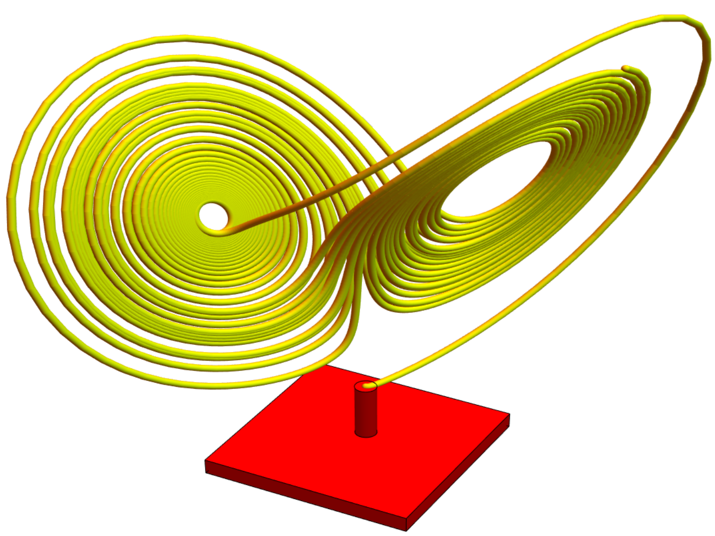


19.8. In mechanics, **Hamiltonian fields** plays an important role: if $H(x, y)$ is a function of two variables called energy, then $[H_y(x, y), -H_x(x, y)]$ is called a **Hamiltonian vector field**. An example is the **harmonic oscillator** $H(x, y) = (x^2 + y^2)/2$. Its vector field is $\vec{F}(x, y) = [H_y(x, y), -H_x(x, y)] = [y, -x]$. The flow lines of a Hamiltonian vector fields are located on the level curves of H.

19.9. Here is a famous example. It is the **Lorenz vector field**

$$\vec{F}(x, y, z) = \begin{bmatrix} 10y - 10x \\ -xz + 28x - y \\ xy - \frac{8}{3}z \end{bmatrix} .$$

It features what one calls a **strange attractor**, an icon in **chaos theory**.



Homework

This homework is due on Tuesday, 7/29/2025.

Problem 19.1:
a) Draw the gradient vector field of $f(x,y) = \sqrt{(x-4)^2 + (y-2)^2}$.
b) Draw the gradient vector field of $f(x,y) = \sin(x^2 - y^2)$.
In both cases, draw a contour map of f and use gradients to draw the vector field $\vec{F}(x,y) = \nabla f(x,y)$.

Problem 19.2: The vector field

$$\vec{F}(x,y) = \begin{bmatrix} \frac{x}{(x^2+y^2)^{(3/2)}} \\ \frac{y}{(x^2+y^2)^{(3/2)}} \end{bmatrix}$$

appears in electrostatics. Find a function $f(x,y)$ such that $\vec{F} = \nabla f$.

Problem 19.3:
a) Is the vector field $\vec{F}(x,y) = \begin{bmatrix} xy \\ x^2 \end{bmatrix}$ a gradient field?
b) Is the vector field $\vec{F}(x,y) = \begin{bmatrix} \sin(x) + y \\ \cos(y) + x \end{bmatrix}$ a gradient field?
In both cases, find $f(x,y)$ satisfying $\nabla f(x,y) = \vec{F}(x,y)$ or give a reason, why it does not exist.

Problem 19.4: Find conditions such that a vector field in three dimensions $\vec{F}(x,y,z)$ is a gradient field. Then check it in the following cases. If the field is a gradient field, find a potential f such that $\vec{F} = \nabla f$.
a) $\vec{F}(x,y,z) = [x^{11}, y^9, z]$.
b) $\vec{F}(x,y,z) = [y, x, z^3]$.
c) $\vec{F}(x,y,z) = [10y + 10x, 10x + 10y, x]$.
d) $\vec{F}(x,y) = [y, z, x]$.

Problem 19.5: Find the potential function $f(x,y,z)$ to

$$\vec{F}(x,y,z) = [5e^{5x} + 5x^4y + z^4 + y\cos(xy), x^5 + x\cos(xy), 4xz^3 + 7e^{7z}] \,.$$

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