MULTIVARIABLE CALCULUS

MATH S-21A

Unit 22: Curl and Flux

Lecture

22.1. In two dimensions, the **curl** of $\vec{F}$ was the **scalar field** $\text{curl}(\vec{F}) = Q_x - P_y$. By Green's theorem, the curl evaluated at (x, y) is $\lim_{r\to 0} \int_{C_r} \vec{F}\ dr/(\pi r^2)$, where C_r is a small circle of radius r oriented counter clockwise an centered at (x, y). Green's theorem explains that curl measures how the field "curls" or rotates. As rotations in two dimensions are determined by a single angle, in three dimensions, three parameters in the form of a vector are needed A direction of this vector tells the axes of rotation and the magnitude tells the amount of rotation.

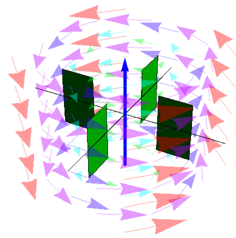
Definition: The **curl** of $\vec{F} = [P, Q, R]$ is the vector field

$$\text{curl}([P, Q, R]) = [R_y - Q_z, P_z - R_x, Q_x - P_y] .$$

22.2. In **Nabla calculus**, this is written as $\text{curl}(\vec{F}) = \nabla \times \vec{F}$. Note that the third component $Q_x - P_y$ of the curl is for fixed z just the curl of the 2-dimensional vector field $\vec{F} = [P, Q]$. While the curl in two dimensions is a scalar field, it is a vector field in 3 dimensions. In n dimensions, it would have $n(n - 1)/2$ components, the number of 2-dimensional coordinate planes. The curl measures the "vorticity" of the field. Each of the components is the rotation projected onto one the three coordinate planes.

Definition: If $\vec{F}$ has zero curl everywhere, the field is called **irrotational**.

22.3. The curl is frequently visualized using a "paddle wheel". If the rotation axes points into direction $\vec{v}$, the signed rotation speed is $\vec{F} \cdot \vec{v}$. If the vector $\vec{v}$ is chosen into the direction so that the wheel turns fastest, this is the direction of $\text{curl}(\vec{F})$. The angular velocity of the wheel represents the magnitude of the curl.



22.4. In two dimensions, we had two derivatives, the gradient and curl. In three dimensions, there are now three fundamental derivatives: the **gradient**, the **curl** and the **divergence**.

Definition: The **divergence** of $\vec{F} = [P, Q, R]$ is the scalar field $\text{div}([P, Q, R]) = \nabla \cdot \vec{F} = P_x + Q_y + R_z$.

22.5. The divergence can also be defined in two dimensions, but it should there be seen differently as the adjoint of the gradient and not an "exterior derivative". It helps to have in n dimensions to have exactly n fundamental derivatives and n exactly fundamental integrals and exactly n fundamental theorems. Distinguishing dimensions helps to organize the integral theorems. While Green looks like Stokes, we urge you to look at it as a different theorem, living in "flatland". It is clearer to keep the calculus confined to its dimension It makes the topics easier to remember.

Definition: In two dimensions, the **divergence** of $\vec{F} = [P, Q]$ is defined as $\text{div}([P, Q]) = \nabla \cdot \vec{F} = P_x + Q_y$. It defines the Laplacian $\Delta f(x, y) = \text{div}(\text{grad}(f) = f_{xx} + f_{yy}$.

22.6. In two dimensions, the divergence as the curl of a -90 degrees rotated field $\vec{G} = [Q, -P]$ because $\text{div}(\vec{G}) = Q_x - P_y = \text{curl}(\vec{F})$ but divergence measures the "expansion" of $\vec{F} = [P, Q]$. Fields with zero divergence are called **incompressible**.

22.7. With the "vector" $\nabla = [\partial_x, \partial_y, \partial_z]$, we can write $\text{curl}(\vec{F}) = \nabla \times \vec{F}$ and $\text{div}(\vec{F}) = \nabla \cdot \vec{F}$. Rewriting formulas using the "Nabla vector" and using rules from geometry gives a **Nabla calculus** which works both in 2 and 3 dimensions. The ∇ vector is not an actual vector but an **operator**. The following combination of divergence and gradient often appears in physics:

Definition: $\Delta f = \text{div}(\text{grad}(f)) = f_{xx} + f_{yy} + f_{zz}$ is the **Laplacian** of $f(x, y, z)$. One can write $\Delta f = \nabla^2 f$ because formally $\nabla \cdot \nabla = \Delta$.

22.8. We can extend the Laplacian also to vector fields by defining:

Definition: $\Delta \vec{F} = [\Delta P, \Delta Q, \Delta R]$ and write $\nabla^2 \vec{F}$.

Δ, the"1-form Laplacian" plays a role in electromagnetism. We do not use it here. Here are some identities: $\text{div}, \text{grad}, \text{curl}, \Delta$.

$\text{div}(\text{curl}(\vec{F})) = 0$ $\qquad$ $\text{curlgrad}(\vec{F}) = \vec{0}$

$\text{curl}(\text{curl}(\vec{F})) = \text{grad}(\text{div}(\vec{F})) - \Delta(\vec{F})$.

EXAMPLES

22.9. Question: Is there a vector field $\vec{G}$ such that $\vec{F} = [x + y, z, y^2] = \text{curl}(\vec{G})$? **Answer:** No, because $\text{div}(\vec{F}) = 1$ is incompatible with $\text{div}(\text{curl}(\vec{G})) = 0$.

22.10. Show that in simply connected region, every irrotational and incompressible field can be written as a vector field $\vec{F} = \text{grad}(f)$ with $\Delta f = 0$. Proof. Since $\vec{F}$ is irrotational, there exists a function f satisfying $F = \text{grad}(f)$. From $\text{div}(F) = 0$, one gets $\text{divgrad}(f) = \Delta f = 0$.

22.11. Here is a remark: If we rotate the vector field $\vec{F} = [P, Q]$ by 90 degrees $= \pi/2$, we get a new vector field $\vec{G} = [-Q, P]$. The integral $\int_C F \cdot ds$ becomes a **flux** $\int_\gamma G \cdot \vec{dn}$ of G through the boundary of R, where $\vec{dn}$ is a normal vector to $\vec{dr}$ with length $dn = dr = |r'|dt$. With $\text{div}(\vec{F}) = (P_x + Q_y)$, we see that

$$\text{curl}(\vec{F}) = \text{div}(\vec{G}) \; .$$

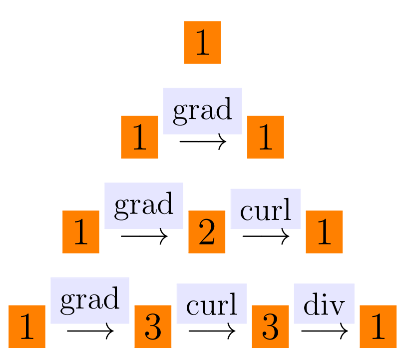
Green's theorem now becomes

$$\int\int_R \text{div}(\vec{G}) \; dxdy = \int_C \vec{G} \cdot \vec{dn} \; ,$$

where $\vec{dn}(x, y)$ is a normal vector at (x, y) orthogonal to the velocity vector $\vec{r}\,'(x, y)$ at (x, y). In three dimensions this theorem will become the divergence theorem. We do not treat this however as a different theorem in two dimensions. It is just Green's theorem in disguise.

In two dimensions, the divergence at a point (x, y) is the average flux of the field through a small circle of radius r around the point in the limit when the radius of the circle goes to zero.

We have now all the derivatives we need. In dimension n, there are n fundamental derivatives.



Homework

This homework is due on Tuesday, 8/5/2025.

Problem 22.1: Construct your own nonzero vector field $\vec{F}(x,y) = [P(x,y), Q(x,y)]$ in each of the following cases:
a) $\vec{F}$ is irrotational and incompressible.
b) $\vec{F}$ is irrotational but not incompressible.
c) $\vec{F}$ is incompressible but not irrotational.
d) $\vec{F}$ is not irrotational and not incompressible.

Problem 22.2: The vector field $\vec{F}(x,y,z) = [x,y,-2z]$ satisfies $\text{div}(\vec{F}) = 0$. Can you find a vector field $\vec{G}(x,y,z)$ such that $\text{curl}(\vec{G}) = \vec{F}$? Such a field $\vec{G}$ is called a **vector potential**.
Hint. Write $\vec{F}$ as a sum $[x,0,-z] + [0,y,-z]$ and find vector potentials for each of the parts using a vector field you have seen in class.

Problem 22.3: Evaluate the flux integral $\iint_S [0,0,yz] \cdot d\vec{S}$, where S is the surface with parametric equation $x = uv, y = u+v, z = u-v$ on $R : u^2+v^2 \leq 4$ and $u > 0$.

Problem 22.4: Evaluate the flux integral $\iint_S \text{curl}(F) \cdot d\vec{S}$ for

$$\vec{F}(x,y,z) = [3xy, 3yz, 3zx] \,.$$

where S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $[0,2] \times [0,2]$ and has an upward orientation.

Problem 22.5: a) What is the relation between the flux of the vector field $\vec{F} = \nabla g/|\nabla g|$ through the surface $S : \{g = 1\}$ with $g(x,y,z) = 5x^6 + y^4 + 2z^3$ and the surface area of S?
b) Find the flux of the vector field $\vec{G} = \nabla g \times [0,0,222]$ through the surface S.

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Oliver Knill, knill@math.harvard.edu, Math S-21a, Harvard Summer School, 2025

[1] Both a and b) do not need any computation. You can answer each question with one sentence. In part a) compare $\vec{F} \cdot d\vec{S}$ with dS in that case.