MULTIVARIABLE CALCULUS

MATH S-21A

Unit 24: Divergence Theorem

Lecture

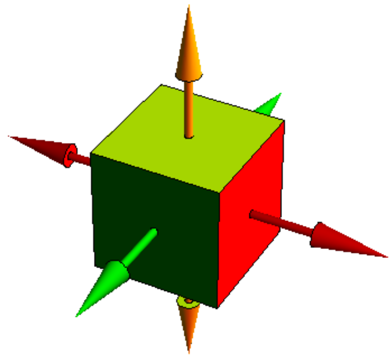
24.1. We have already seen the **fundamental theorem of line integrals** and **Stokes theorem**. The **divergence theorem** [1] completes the list of integral theorems in three dimensions:

Theorem: Divergence Theorem. If E be a solid bounded by a surface S. The surface S is oriented so that the normal vector points outside. If $\vec{F}$ be a vector field, then

$$\iiint_E \operatorname{div}(\vec{F})\, dV = \iint_S \vec{F} \cdot d\vec{S} \ .$$

24.2. To see why this is true, take a small box $[x, x+dx] \times [y, y+dy] \times [z, z+dz]$. The flux of $\vec{F} = [P, Q, R]$ through the faces perpendicular to the x-axes is $[\vec{F}(x+dx, y, z) \cdot [1, 0, 0] + \vec{F}(x, y, z) \cdot [-1, 0, 0]]dydz = (P(x+dx, y, z) - P(x, y, z))dydz \sim P_x \ dxdydz$. Similarly, the flux through the y-boundaries is $P_y \ dydxdz$ and the flux through the two z-boundaries is $P_z \ dzdxdy$. The total flux through the faces of the cube is $(P_x + Q_y + R_z) \ dxdydz = \operatorname{div}(\vec{F}) \ dxdydz$. A general solid E can be approximated as a union of small cubes. The sum of the fluxes through all the cubes consists now of the flux through all faces without neighboring faces. The fluxes through adjacent sides cancel and only the flux through the boundary survives. The sum of all the $\operatorname{div}(\vec{F}) \ dxdydz$ is a Riemann sum approximation for the integral $\iiint_G \operatorname{div}(\vec{F}) \ dxdydz$. In the limit, when dx, dy, dz all go to zero, we obtain the divergence theorem.

[1]Also called Gauss Theorem



24.3. The theorem explains what divergence means. If we integrate the divergence over a small cube, it is equal the flux of the field through the boundary of the cube. If this is positive, then more field exits the cube than entering the cube. There is field "generated" inside. The divergence measures the "expansion" of the field.

Examples

24.4. Let $\vec{F}(x,y,z) = [x,y,z]$ and let S be the unit sphere. The divergence of $\vec{F}$ is the constant function $\mathrm{div}(\vec{F}) = 3$ and $\iiint_G \mathrm{div}(\vec{F})\; dV = 3 \cdot 4\pi/3 = 4\pi$. The flux through the boundary is $\iint_S \vec{r} \cdot \vec{r}_u \times \vec{r}_v \; dudv = \iint_S |\vec{r}(u,v)|^2 \sin(v)\; dudv = \int_0^{\pi} \int_0^{2\pi} \sin(v)\; dudv = 4\pi$ also. We see that the divergence theorem allows us to compute the area of the sphere from the volume of the enclosed ball or compute the volume from the surface area.

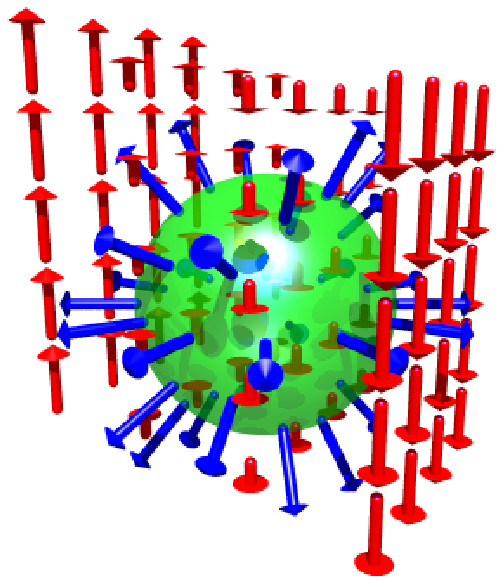
24.5. What is the flux of the vector field $\vec{F}(x,y,z) = [2x, 3z^2 + y, \sin(x)]$ through the solid $G = [0,3]\times[0,3]\times[0,3] \setminus ([0,3]\times[1,2]\times[1,2] \cup [1,2]\times[0,3]\times[1,2] \cup [0,3]\times[0,3]\times[1,2])$ which is a cube where three perpendicular cubic holes have been removed? **Solution:** Use the divergence theorem: $\mathrm{div}(\vec{F}) = 2$ and so $\iiint_G \mathrm{div}(\vec{F})\; dV = 2\iiint_G \; dV = 2\mathrm{Vol}(G) = 2(27-7) = 40$. Note that the flux integral here would be over a complicated surface over dozens of rectangular planar regions.

24.6. Find the flux of $\mathrm{curl}(\vec{F})$ through a torus if $\vec{F} = [yz^2, z + \sin(x) + y, \cos(x)]$ and the torus has the parametrization

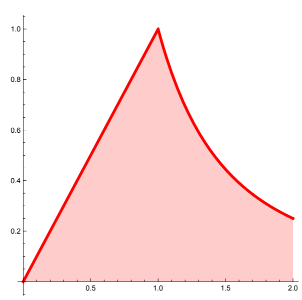
$$\vec{r}(\theta,\phi) = [(2+\cos(\phi))\cos(\theta), (2+\cos(\phi))\sin(\theta), \sin(\phi)] \; .$$

Solution: The answer is 0 because the divergence of $\mathrm{curl}(\vec{F})$ is zero. By the divergence theorem, the flux is zero.

24.7. Similarly as Green's theorem allowed us to calculate the area of a region by passing along the boundary, the volume of a region can be computed as a flux integral: Take for example the vector field $\vec{F}(x,y,z) = [x,0,0]$ which has divergence 1. The flux of this vector field through the boundary of a solid region is equal to the volume of the solid: $\iint_{\delta G} [x,0,0] \cdot d\vec{S} = \text{Vol}(G)$.



24.8. How heavy are we at distance r from the center of the earth?
Solution: The law of gravity can be formulated as $\text{div}(\vec{F}) = 4\pi\rho$, where ρ is the mass density. We assume that the earth is a ball of radius R. By rotational symmetry, the gravitational force is normal to the surface: $\vec{F}(\vec{x}) = \vec{F}(r)\vec{x}/||\vec{x}||$. The flux of $\vec{F}$ through a ball of radius r is $\iint_{S_r} \vec{F}(x) \cdot d\vec{S} = 4\pi r^2 \vec{F}(r)$. By the **divergence theorem**, this is $4\pi M_r = 4\pi \iiint_{B_r} \rho(x)\ dV$, where M_r is the mass of the material inside S_r. We have $(4\pi)^2 \rho r^3/3 = 4\pi r^2 \vec{F}(r)$ for $r < R$ and $(4\pi)^2 \rho R^3/3 = 4\pi r^2 \vec{F}(r)$ for $r \geq R$. Inside the earth, the gravitational force $\vec{F}(r) = 4\pi\rho r/3$. Outside the earth, it satisfies $\vec{F}(r) = M/r^2$ with $M = 4\pi R^3 \rho/3$.



24.9. We conclude with an overview over the integral theorems and give an other typical example in each case.

The fundamental theorem for line integrals, Green's theorem, Stokes theorem and divergence theorem are all part of **one single theorem** $\int_G dF = \int_{\delta G} F$, where dF is a **exterior derivative** of a **field** F and where δG is the **boundary** of a **geometry** G. It generalizes the fundamental theorem of calculus.

Fundamental theorem of line integrals: If C is a curve with boundary $\{A, B\}$ and f is a function, then
$$\int_C \nabla f \cdot \vec{dr} = f(B) - f(A)$$

24.10. Remarks.

1) For closed curves, the line integral $\int_C \nabla f \cdot \vec{dr}$ is zero.
2) Gradient fields are **path independent**: if $\vec{F} = \nabla f$, then the line integral between two points P and Q does not depend on the path connecting the two points.
3) The line integral theorem holds in any dimension. In one dimension, it reduces to the **fundamental theorem of calculus** $\int_a^b f'(x)\, dx = f(b) - f(a)$
4) The theorem justifies the name **conservative** for gradient vector fields.
5) The term "potential" was coined by George Green who lived from 1783-1841.

24.11. Example. Let $f(x, y, z) = x^2 + y^4 + z$. Find the line integral of the vector field $\vec{F}(x, y, z) = \nabla f(x, y, z)$ along the path $\vec{r}(t) = [\cos(5t), \sin(2t), t^2]$ from $t = 0$ to $t = 2\pi$.

Solution. $\vec{r}(0) = [1, 0, 0]$ and $\vec{r}(2\pi) = [1, 0, 4\pi^2]$ and $f(\vec{r}(0)) = 1$ and $f(\vec{r}(2\pi)) = 1 + 4\pi^2$. The fundamental theorem of line integral gives $\int_C \nabla f\ \vec{dr} = f(r(2\pi)) - f(r(0)) = 4\pi^2$.

Green's theorem. If R is a region with boundary C and $\vec{F}$ is a vector field, then
$$\iint_R \text{curl}(\vec{F})\ dxdy = \int_C \vec{F} \cdot \vec{dr}\ .$$

24.12. Remarks.
1) Greens theorem allows to switch from double integrals to one-dimensional integrals.
2) The curve is oriented in such a way that the region is to the left.
3) The boundary of the curve can consist of piecewise smooth pieces.
4) If $C : t \mapsto \vec{r}(t) = [x(t), y(t)]$, the line integral is $\int_a^b [P(x(t), y(t)), Q(x(t), y(t))] \cdot [x'(t), y'(t)]\ dt$.
5) Green's theorem was found by George Green (1793-1841) in 1827 and by Mikhail Ostrogradski (1801-1862).
6) If $\text{curl}(\vec{F}) = 0$ in a simply connected region, then the line integral along a closed curve is zero. If two curves connect two points then the line integral along those curves

agrees.
7) Use $\vec{F}(x,y) = [-y,0]$ or $\vec{F}(x,y) = [0,x]$ to compute **area**.

24.13. Example. Find the line integral of the vector field $\vec{F}(x,y) = [x^4 + \sin(x) + y, x + y^3]$ along the path $\vec{r}(t) = [\cos(t), 5\sin(t) + \log(1 + \sin(t))]$, where t runs from $t = 0$ to $t = \pi/2$.

Solution. $\text{curl}(\vec{F}) = 0$ implies that the line integral depends only on the end points $(1,0), (0,5)$ of the path. Take the simpler path $\vec{r}(t) = [-t, 0], -1 \leq t \leq 1$, which has velocity $\vec{r}\,'(t) = [-1, 0]$. The line integral is $\int_{-1}^{1} [t^4 - \sin(t), -t] \cdot [-1, 0]\, dt = -t^5/5|_{-1}^{1} = -2/5$.

Remark We could also find a potential $f(x,y) = x^5/5 - \cos(x) + xy + y^5/4$. It has the property that $\text{grad}(f) = \vec{F}$. Again, we get $f(0,-1) - f(0,1) = -1/5 - 1/5 = -2/5$.

Stokes theorem. If S is a surface with boundary C and $\vec{F}$ is a vector field, then

$$\int\int_S \text{curl}(\vec{F}) \cdot dS = \int_C \vec{F} \cdot d\vec{r} \ .$$

24.14. Remarks.
1) Stoke's theorem can imply Green's theorem: if $\vec{F}$ is z-independent and the surface S is contained in the xy-plane, one obtains the result of Green.
2) The orientation of C is such that if you walk along C and have your head in the direction of the normal vector $\vec{r}_u \times \vec{r}_v$, then the surface to your left.
3) Stokes theorem was also found by André Ampère (1775-1836) in 1825 and by George Stokes (1819-1903).
4) The flux of the curl of a vector field does not depend on the surface S, only on the boundary of S.
5) The flux of the curl through a closed surface like the sphere is zero: the boundary of such a surface is empty.

24.15. Example. Compute the line integral of $\vec{F}(x,y,z) = [x^3 + xy, y, z]$ along the polygonal path C connecting the points $(0,0,0), (2,0,0), (2,1,0), (0,1,0)$ in that order.
Solution. The path C bounds a surface $S : \vec{r}(u,v) = [u,v,0]$ parameterized by $R = [0,2] \times [0,1]$. By Stokes theorem, the line integral is equal to the flux of $\text{curl}(\vec{F})(x,y,z) = [0,0,-x]$ through S. The normal vector of S is $\vec{r}_u \times \vec{r}_v = [1,0,0] \times [0,1,0] = [0,0,1]$ so that $\int\int_S \text{curl}(\vec{F})\, d\vec{S} = \int_0^2 \int_0^1 [0,0,-u] \cdot [0,0,1]\, dudv = \int_0^2 \int_0^1 -u\, dudv = -2$.

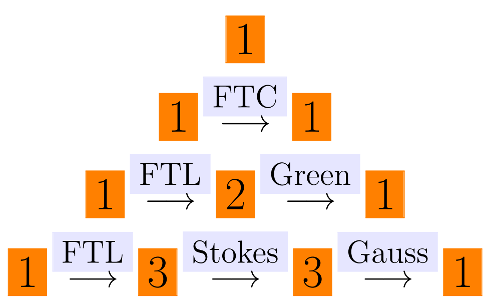
Divergence theorem: If S is the boundary of a region E in space and $\vec{F}$ is a vector field, then

$$\iiint_E \operatorname{div}(\vec{F})\, dV = \iint_S \vec{F} \cdot d\vec{S}\ .$$

24.16. Remarks.
1) The divergence theorem is also called **Gauss theorem**.
2) It is useful to determine the flux of vector fields through surfaces.
3) Gauss theorem can be used to compute volume.
4) The theorem was also found 1764 by Joseph Louis Lagrange (1736-1813), and by Carl Friedrich Gauss (1777-1855) as well as by George Green.
5) For divergence free vector fields $\vec{F}$, the flux through a closed surface is zero. Such fields $\vec{F}$ are also called **incompressible** or **source free**.

24.17. Example. Compute the flux of the vector field $\vec{F}(x,y,z) = [-x, y, z^2]$ through the boundary S of the rectangular box $[0,3] \times [-1,2] \times [1,2]$.
Solution. By Gauss theorem, the flux is equal to the triple integral of $\operatorname{div}(F) = 2z$ over the box: $\int_0^3 \int_{-1}^2 \int_1^2 2z\ dxdydz = (3-0)(2-(-1))(4-1) = 27$.

24.18. How do these theorems fit together? In n-dimensions, there are n theorems. We have here seen the situation in dimension $n = 2$ and $n = 3$, but one could continue. The fundamental theorem of line integrals generalizes directly to higher dimensions. Also the divergence theorem generalizes directly to n dimensions. The generalization of curl and flux would need more explanation as for $n = 4$ already, the curl of a vector field is a 6-dimensional object. It is a $n(n-1)/2$ dimensional object called a 2-form.



24.19. In one dimension, there is one derivative: $f(x) \to f'(x)$. It maps scalar functions to scalar functions. It corresponds to the entry $1-1$ in the Pascal triangle. The next entry $1-2-1$ corresponds to differentiation in two dimensions, where we have the gradient $f \to \nabla f$ mapping a scalar function to a vector field with 2 components as well as the curl, $F \to \operatorname{curl}(F)$ which corresponds to the transition $2-1$ going back to a scalar function. The situation in three dimensions is captured by the entry $1-3-3-1$ in the Pascal triangle. The first derivative $1-3$ is the gradient. The second derivative $3-3$ is the curl and the third derivative $3-1$ is the divergence which gives again a scalar. In $n = 4$ dimensions, we would have to look at $1-4-6-4-1$. The first derivative $1-4$ is still the gradient. Then we have a first curl, which maps a vector field with 4 components into an object with 6 components. Then there is a second curl, which maps an object with 6 components back to a vector field, we would have to look at $1-4-6-4-1$.

24.20. When setting up calculus in dimension n, one talks about **differential forms** instead of scalar fields or vector fields. Functions are 0 forms (scalars) or n-forms (pseudo scalars). Vector fields can be described by 1-forms or $(n-1)$-forms (pseudo vector fields). The general formalism defines a derivative d called **exterior derivative** on differential forms. It maps k-forms to $(k+1)$-forms. There is also an integration of k-forms on k-dimensional objects. The **boundary operation** δ which maps a k-dimensional object into a $(k-1)$- dimensional object. This boundary operation is dual to differentiation. They both satisfy the same relation $dd(F)=0$ and $\delta\delta G=0$. Differentiation and integration are linked by the general Stokes theorem for **G**eometry G and **F**ield F:

$$\int_{\delta G} F = \int_G dF$$

24.21. One can see this as a single theorem, the **fundamental theorem of multivariable calculus**. The theorem is simpler in quantum calculus, where geometric objects and fields are on the same footing. There are various ways how one can generalize this. One way is to write it as $< \delta G, F >=< G, dF >$ which in linear algebra would be written as $[A^T v, w] = [v, Aw]$, where A^T is the transpose of a matrix A $[v,w]$ is the dot product. Since traditional calculus we deal with "smooth" functions and fields, we have to pay a price and consider in turn "singular" objects like points or curves and surfaces. These are idealized objects which have zero diameter, radius or thickness. Modern mathematics also generalizes this in the form of **currents**, objects dual to differential forms.

24.22. So, it is all about **geometries** and **fields**. **Geometries** are curves, or surfaces or solids. **Fields** are scalar functions or vector fields. Geometries G can be "differentiated" by taking the boundary δG. Fields F can be differentiated by applying differential operators dF like grad, curl or div. And then there is integration which pairs up geometries G and fields F. The fundamental theorem $\int_{\delta G} F = \int_G dF$ tells that taking the boundary on the object is dual to taking the derivative of the field.

24.23. Nature likes simplicity and elegance [2] and therefore found a quantum mathematics to be more fundamental. But a geometry in which **geometries and fields are indistinguishable** manifests only in the very small.

[2] Leibniz: 1646-1716

Homework

This homework is due on Tuesday, 8/5/2025.

Problem 24.1: What is the flux of the vector field $\vec{F}(x,y,z) = [9y, 2xy, 4yz + 234xy]$ through the unit cube $[0,1] \times [0,1] \times [0,1]$.

Problem 24.2: Find the flux of the vector field $\vec{F}(x,y,z) = [xy, yz, zx]$ through the cylinder $x^2 + y^2 \leq 1$, $0 \leq z \leq 2$ without the bottom disk $z = 0$. Hint: close the surface first and then also find the flux through the bottom.

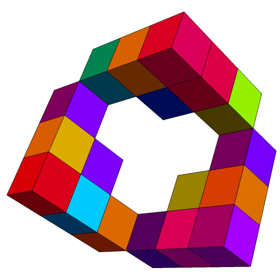
Problem 24.3: Use the divergence theorem to calculate the flux of $\vec{F}(x,y,z) = [x^3, y^3, z^3]$ through the sphere $S : x^2 + y^2 + z^2 = 1$, where the sphere is oriented so that the normal vector points outwards.

Problem 24.4: Assume the vector field

$$\vec{F}(x,y,z) = [15x^3 + 42xy^2, y^3 + e^y \sin(z), e^y \cos(z) + 15z^3]$$

is the magnetic field of the **sun** whose surface is a sphere of radius 3 oriented with the outward orientation. Compute the magnetic flux $\iint_S \vec{F} \cdot d\vec{S}$.

Problem 24.5: Find $\int\int_S \vec{F} \cdot d\vec{S}$, where $\vec{F}(x,y,z) = [-150x, 25y, 25z]$ and S is the boundary of the solid built with the 18 cubes shown in the picture. Each cube has length 1.



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