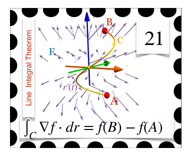
Triple Integrals

- $\iiint_E f(x,y,z)\ dzdydx = \iiint_E f\ dV$ triple integral
- $\int_a^b \int_c^d \int_u^v f(x,y,z)\ dzdydx$ integral over a cuboid, a rectangular box
- $\int_a^b [\iint_{G(z)} f(x,y,z)\ dA] dz$ hamburger cut, $G(z)$ =salad, cheese, tomato, beef slice
- $\iint_G [\int_{a(x,y)}^{b(x,y)} f(x,y,z)\ dz] dydx$ french fries cut, the inner integral are the fries
- $\iiint_E f(r,\theta,z) \boxed{r}\ dzdrd\theta$ integral in cylindrical coordinates
- $\iiint_E f(\rho,\theta,\phi) \boxed{\rho^2 \sin(\phi)}\ d\rho d\phi d\theta$ integral in spherical coordinates
- $\int_a^b \int_c^d \int_u^v f(x,y,z)\ dzdydx = \int_u^v \int_c^d \int_a^b f(x,y,z)\ dxdydz$ Fubini
- $\iiint_E \boxed{1}\ dzdydx = v(E)$ is the volume of solid E
- $\iiint_E \sigma(x,y,z)\ dV$ mass of solid E with density σ

Line Integrals

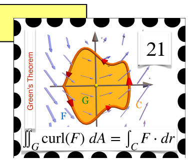
- $\vec{F}(x,y) = [P(x,y), Q(x,y)], \vec{F}(x,y,z) = [P(x,y,z), Q(x,y,z), R(x,y,z)]$ vector field.
- $\int_C \vec{F} \cdot \vec{dr} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$ line integral
- $\vec{F}(x,y) = \nabla f(x,y)$ gradient field = potential field = conservative field

Fundamental theorem of line integrals



- FTL: $\vec{F}(x,y) = \nabla f(x,y)$, $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t))\ dt = f(\vec{r}(b)) - f(\vec{r}(a))$
- Closed loop property $\int_C \vec{F}\ \vec{dr} = 0$ for all closed curves C
- Equivalent: closed loop property, path independence and gradient field
- Clairaut test: $\text{curl}(\vec{F}) \neq 0$ assures $\vec{F}$ is not a gradient field
- If simply connected domain: $\text{curl}(\vec{F}) = 0$ everywhere $\Rightarrow \vec{F} = \nabla f$

Green's Theorem

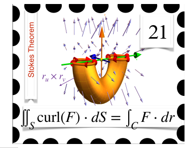


- $\vec{F}(x,y) = [P,Q]$, curl in two dimensions: $\text{curl}(\vec{F}) = Q_x - P_y$
- Green's theorem: C boundary of G, then $\int_C \vec{F} \cdot dr = \iint_G \text{curl}(\vec{F})\ dA$
- Take $\vec{F} = [-y, 0]$ or $\vec{F} = [0, x]$ to get area
- Green's theorem: compute difficult line integrals or difficult 2D integrals
- Orientation: the region G is to the left

Flux integrals

- $\vec{F}(x,y,z)$ vector field, $S = \vec{r}(G)$ parametrized surface
- $\vec{r}_u \times \vec{r}_v$ normal vector, $\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ unit normal vector
- $\vec{r}_u \times \vec{r}_v\ dudv = \vec{dS} = \vec{n}\ dS$ normal surface element
- $\iint_S \vec{F} \cdot \vec{dS} = \iint_S \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dudv$ flux integral
- Orientation: the flux changes if the orientation of the surface is reversed

Stokes Theorem

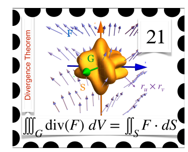


$\vec{F}(x,y,z) = [P,Q,R]$, $\text{curl}([P,Q,R]) = [R_y - Q_z, P_z - R_x, Q_x - P_y] = \nabla \times \vec{F}$
Stokes's theorem: C boundary of surface S, then $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$
Stokes theorem: compute difficult flux integrals or difficult line integrals
Orientation: walk on boundary curve with surface "to the left"

Grad Curl Div

$\nabla = [\partial_x, \partial_y, \partial_z]$, $\vec{F} = \nabla f$, $\text{curl}(\vec{F}) = \nabla \times \vec{F}$, $\text{div}(\vec{F}) = \nabla \cdot \vec{F}$
$\text{div}(\text{curl}(\vec{F})) = 0$ and $\text{curl}(\text{grad}(f)) = \vec{0}$
$\text{div}(\text{grad}(f)) = \Delta f$ Laplacian
incompressible = divergence free: $\text{div}(\vec{F}) = 0$ everywhere. Implies $\vec{F} = \text{curl}(\vec{G})$
irrotational = $\text{curl}(\vec{F}) = 0$ everywhere. Implies $\vec{F} = \text{grad}(f)$

Divergence Theorem



$\text{div}([P,Q,R]) = P_x + Q_y + R_z = \nabla \cdot \vec{F}$
divergence theorem: solid E, boundary S then $\iint_S \vec{F} \cdot dS = \iiint_E \text{div}(\vec{F})\, dV$
divergence theorem: use to compute difficult flux or 3D integrals

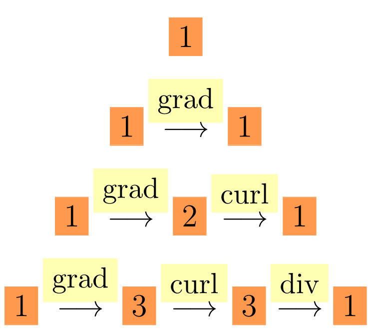
Some topology

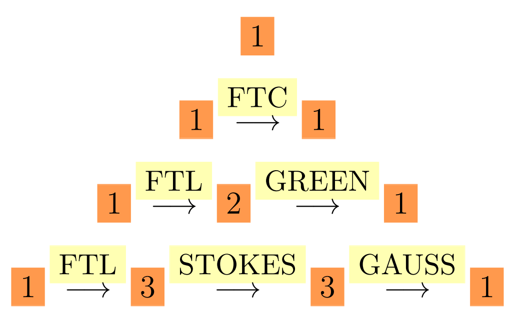
interior of region E: points in E for which small neighborhood is still in E
boundary of curve: the end points of the curve if they exist
boundary of surface S: points on S not in the interior: Example: rim of disc
boundary of solid E: part of the solid not in the interior of E
closed surface: a surface without boundary. Examples: sphere, torus
closed curve: a curve with no boundary. Examples: circle, knot.

Integration overview

Line integral: $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$
Flux integral: $\iint_S \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dudv$

Theorems and Derivatives overview





Oliver Knill, Math S 21a, last update: August 3, 2025