SECOND HOURLY in 21 TWEETS **Maths 21a 2014-2025**

Each tweet (X-Post) **140** characters or less.

Cairaut: $f_{xy} = f_{yx}$ Proof: take $h \to 0$ limit of
$f_{xy} \sim h^2[(f(x+h, y+h) - f(x, y+h)) - (f(x+h, y) - f(x, y))]$
$f_{yx} \sim h^2[(f(x+h, y+h) - f(x+h, y)) - (f(x, y+h) - f(x, y))]$

ODE: equation for function f involving derivatives
PDE: equation for function f involving partial derivatives. Example: $f_x = f_{yy}f + f^2$

Heat equation: $u_t = u_{xx}$
Wave equation: $u_{tt} = u_{xx}$
Laplace equation $u_{xx} + y_{yy} = 0$
Transport equation $u_t = u_x$
Burgers equation $u_t + uu_x = u_{xx}$

Gradients are perpendicular to level sets.
Proof: $\vec{r}(t)$ on $f = c$ satisfies
$0 = d/dt f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$
showing that ∇f is orthogonal

Directional derivative maximizes in gradient direction. Proof:
$D_{\vec{v}} f = |\nabla f \cdot \vec{v}| = |\nabla f| \cos(a)$
For $\vec{w} = \nabla f / |\nabla f|$,
$D_{\vec{w}} f = |\nabla f|$

Second derivative test: discriminant
$D = f_{xx}f_{yy} - f_{xy}^2$ and $A = f_{xx}$ determine:
$D > 0, A > 0 \Rightarrow$ min, $D > 0, A < 0 \Rightarrow$ max,
$D < 0 \Rightarrow$ saddle, $D = 0$: not know.

Lagrange:
$f_x = \lambda g_x$
$f_y = \lambda g_y$
$g(x, y) = c$
Proof: $\nabla(f)$ and $\nabla(g)$ are parallel. Else moving on $g = c$ crosses level curves.

Chain rule $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$. Implicit differentiation $f(x, g(x)) = 0$ implies $g'(x) = -f_x/f_y$.

Tangent plane at P : find $\nabla f = [a, b, c]$ and plane $ax + by + cz = d$, (get d by plugging in P). $\nabla f = [a, b]$ gives tangent line $ax + by = d$.

Estimate $f(3.001, 4.9999)$ by computing the gradient $[a, b]$ of f at $(3, 5)$ and get $L(3.001, 4.9999) = f(3, 5) + a \cdot 0.001 - b \cdot 0.0001$.

"Bottom to top" integration on $[a, b]$ on x-axis
"Left to right" integration on $[c, d]$ on y-axis

Double integral $\int\int_R f(x, y)\ dxdy$ interpretation: signed volume under the graph of f. It is a volume if $f \geq 0$.

Fubini: for rectangular regions only:
$\int_a^b \int_c^d f(x, y) dxdy = \int_c^d \int_a^b f(x, y) dydx$

Surface area of a parametrized surface $\vec{r}(u, v)$, defined for a region R is $\int\int_R |\vec{r}_u \times \vec{r}_v| dudv$.

Polar integration:
include an integration factor r.
Proof: $\vec{r}(s, t) = [r\cos(t), r\sin(t), 0]$, $|\vec{r}_r \times \vec{r}_t| = r$.

By parts: $\int u dv = uv - \int v du$
Proof: integrate $uv' + vu' = (uv)'$
Example: $\int x\cos(x)dx = x\sin(x) - \int 1 \cdot \sin(x)dx = x\sin(x) + \cos(x) + C$

Substitution: Example:
$\int x^4 \exp(x^5)dx$
$u = x^5, du = 5x^4dx$ $\int \exp(u)/5du = \exp(x^5)/5 + C$

Tips for double integrals: make picture, consider other coordinates or change order of integration.

Helpful identities:
$\cos^2(t) + \sin^2(t) = 1$
$\cos^2(t) = (1 + \cos(2t))/2$
$\sin^2(t) = (1 - \cos(2t))/2$
$\sin(t)\cos(t) = \sin(2t)/2$

$\int x^n dx = x^{n+1}/(n + 1)$
$\int \exp(ax)dx = \exp(ax)/a$
$\int \cos(ax) = \sin(ax)/a$
$\int \sin(ax) = -\cos(ax)/a$
$\int dx/x = \log(x)$
$\int dx/(1 + x^2) = \arctan(x)$

Know integration by parts, substitution, basic integrals!

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