7/10/2025 FIRST HOURLY **Maths 21a, O.Knill, Summer 2025**

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed. Fill in the boxes

1) T **F** The range of the function $f(x, y) = \log(1 + x^2 + y^2)$ is the entire real axis.

Solution:

False, as the function is always non-negative. The range is $[0, \infty)$.

2) T **F** $(\vec{v} + \vec{w}) \times (\vec{v} - \vec{w}) = \vec{0}$ for all unit vectors $\vec{v}$ and $\vec{w}$.

Solution:

No

3) **T** F If the unit tangent vector $\vec{T}(t)$ of a curve $\vec{r}(t)$ is constant as a function of t, then the curvature $\kappa(t)$ of the curve is constant zero

Solution:

By the definitions $|T'|/|r'|$ of curvature

4) **T** F If the projection $\vec{P}_{\vec{w}}(\vec{v})$ is parallel to $\vec{v}$ then $\vec{v}$ and $\vec{w}$ are parallel.

Solution:

The projection is independent of the length of $\vec{w}$.

5) **T** F If the vectors $\vec{u}, \vec{v}, \vec{w}$ have integer components, then the volume of the parallel-epiped spanned by $\vec{u}, \vec{v}, \vec{w}$ is an integer.

Solution:

The formula shows this. We only multiply integers.

6) **T** F If $\vec{u}, \vec{v}$ are parallel then $\vec{u} + \vec{v}, \vec{u} - \vec{v}$ are parallel.

Solution:

All vectors are parallel to $\vec{u}$.

7) T **F** The identity $|\vec{v} \times \vec{w}| \leq |\vec{v} \cdot \vec{w}|$ holds for all vectors $\vec{v}, \vec{w}$.

Solution:

This is only true if $\sin(\alpha) \leq \cos(\alpha)$

8) **T** F If $\vec{r}(t)$ is a helix, then its torsion is constant. (We mentioned this in class)

Solution:

It is actually both ways, if curvature and torsion are both constant we have a helix.

9) T **F** It is possible to intersect an ellipsoid with a hyperboloid to get an parabola.

Solution:

The intersection is bounded. A parabola is not bounded.

10) **T** F The set of points in space which satisfy $z^2 - y^2 - 1 = -x^2$ is a one-sheeted hyperboloid.

Solution:

It is indeed one-sheeted.

11) T **F** The projection satisfies $\vec{P}_{2\vec{w}}\vec{v} = 2\vec{P}_{\vec{w}}\vec{v}$ for all vectors $\vec{v}$ and $\vec{w}$.

Solution:

The length of the vector $\vec{w}$ does not matter.

12) | T | F | If $\vec{v}(t), \vec{w}(t)$ parametrize curves in space, $\frac{d}{dt}(\vec{v} \times \vec{w})$ is $(\frac{d}{dt}\vec{v}) \times \vec{w} - (\frac{d}{dt}\vec{w}) \times \vec{v}$.

Solution:

This follows from the product rule and the fact that the cross product is anti-commutative.

13) | T | F | The surface given in spherical coordinates as $\sin(\phi) = \rho\cos^2(\phi)$ is a paraboloid.

Solution:

It translates as $r^3 = 1$ meaning $r = 1$.

14) | T | F | If A, B, C, D are four different points space then $\vec{AB} \times \vec{AD}$ is never zero.

Solution:

It can be zero if the vectors are parallel

15) | T | F | The line $\vec{r}(t) = [3, -3, t]$ and the line $\vec{s}(t) = [t, -t, 3]$ intersect and hit each other perpendicularly at this point.

Solution:

The lines go through (0,0,0) and form a right angle.

16) | T | F | The curve given in polar coordinates as $r\cos^2(\theta) = 2\sin(\theta)$ is a parabola.

Solution:

Indeed, $x^2 = 2y$.

17) | T | F | If in spherical coordinates a point is given by $(\rho, \theta, \phi) = (1, 3\pi/2, 0)$, then its rectangular Euclidean coordinates are $(x, y, z) = (0, 0, 1)$.

Solution:

18) T **F** The distance of the points given in polar coordinates as $(r, \theta) = (2, \pi/2)$ and $(r, \theta) = (2, 3\pi/2)$ is equal to $2\sqrt{2}$.

Solution:

It is 4. These are the points $(0, 2)$ and $(0, -2)$.

19) **T** F The surface given in spherical coordinates as $\rho \cos(\phi) = \rho^2$ is sphere.

Solution:

Complete the square.

20) T **F** If $\vec{u}, \vec{v}$ form an obtuse angle and $\vec{v}, \vec{w}$ form an obtuse angle, then $\vec{u}, \vec{w}$ form an obtuse angle.

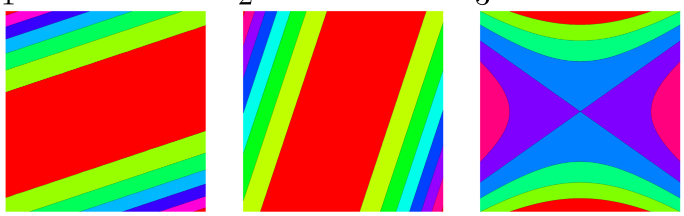
Solution:

Take three vectors in the same plane for which the angles are close to 90, then the other angle is close to 180.

Problem 2) (10 points) No justifications. 0,1,2,3 appear each once in a),b),c),d),e)

a) (2 points) Match functions g with their xy-contour plots. Enter 0 if there is no match.

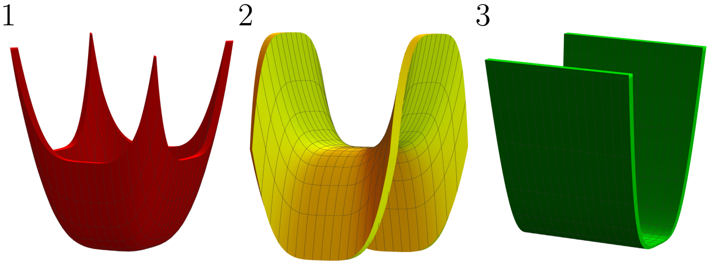
1 2 3



Function $g(x,y) =$	0,1,2, or 3
$x^2 + 3y^2$	
$3x^2 - y^2$	
$(x-3y)^2$	
$(3x-y)^2$	

b) (2 points) Match the graphs of the functions $f(x,y)$. Enter 0 if there is no match.

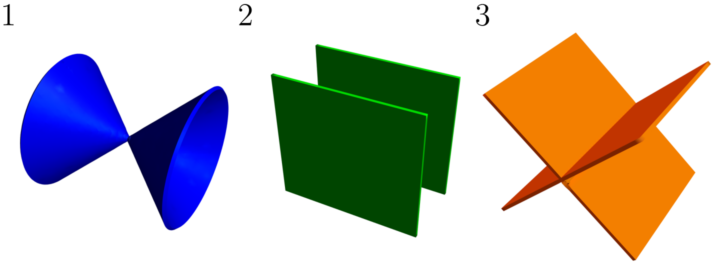
1 2 3



Function $f(x,y) =$	0,1,2, or 3
$\|x\| - \|y\|$	
$x^4 - y^4$	
y^4	
$x^4 + y^4$	

c) (2 points) Match the surfaces $g(x,y,z) = c$. Enter 0 if there is no match.

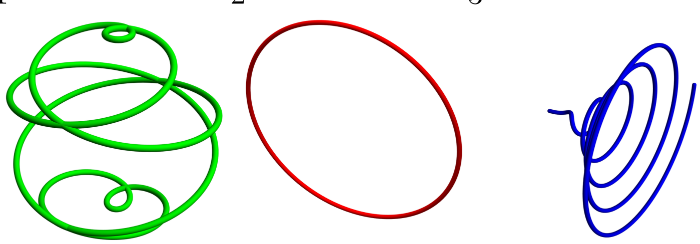
1 2 3



Function $g(x,y,z) =$	0,1,2, or 3
$y^2 = 1$	
$x^2 - z^2 = 0$	
$x^2 + y^2 - z^2 = 1$	
$x^2 - z^2 = y^2$	

d) (2 points) Match the space curves with the parametrizations. Enter 0 if there is no match.

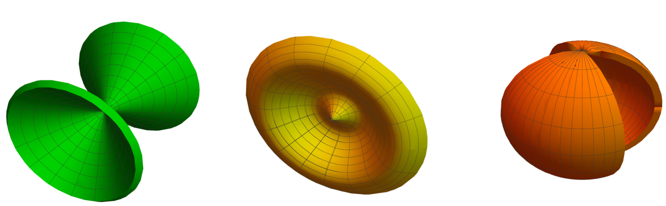
1 2 3



Parametrization $\vec{r}(t) =$	0,1,2, or 3
$[\cos(t^2), 4, \sin(t^2)]$	
$[\sin(t)\cos(5t), \sin(t)\sin(5t), \cos(t)]$	
$[\sin(2t), \cos(t), \sin(t)]$	
$[\cos(t), \cos(t), \cos(t)]$	

e) (2 points) Match the parametrized surfaces. Enter 0 if there is no match.

1 2 3



Parametrization $\vec{r}(u,v) =$	0,1,2, or 3
$[u\cos(v), u, u\sin(v)]$	
$[u\cos(v), v, u\sin(v)]$	
$[u\cos(v), \sin(u), u\sin(v)]$	
$[\sin(u)\cos(v), \cos(u)\cos(v), \sin(v)]$	

Solution:

a) 0312, b) 0231 c) 2301 d) 2130 e) 1023

Problem 3) (10 points) All computations need to be shown.

Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

a) (2 points) Find $(\vec{u} + \vec{v}) \cdot \vec{w}$.

Answer:

b) (2 points) Compute the cos of the angle between $\vec{u}$ and $\vec{v}$ using the dot product.

Answer:

c) (2 points) Compute the sin of the angle between $\vec{v}$ and $\vec{w}$ without dot product.

Answer:

d) (2 points) What is the triple scalar product of the three vectors $\vec{u}, \vec{v}, \vec{w}$?

Answer:

e) (2 points) Compute the triple vector product $\vec{u} \times (\vec{v} \times \vec{w})$.

Answer:

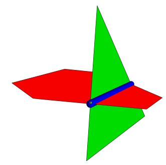
A remark to the last example: the cross product is an operation like division or exponentiation that is non-associative: $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$. You can easily get counter examples $(u/v)/w \neq u/(v/w)$ and $(u^v)^w \neq u^{(v^w)}$, which shows the need to put brackets!

Solution:

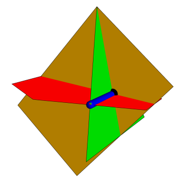
a) 16, b) 8/9, c) $|\vec{v} \times \vec{w}|/(|\vec{v}||\vec{w}|) = \sqrt{17}/9$ d) 5 e) $[0, -8, 8]$.

Problem 4) (10 points)

a) (5 points) Parametrize the line that is the intersection of the two planes $x+y+z=3$ and $x-y-z=-1$.



b) (5 points) Give the equation of the plane through the common point $P=(1,1,1)$ that is perpendicular to the two planes given in a).

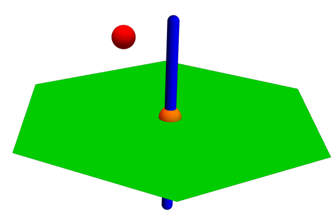


Solution:

a) The vector $[1,1,1]x[1,-1,-1] = [0,2,-2]$ is in the line. The parametrization is $x,y,z = [1,1,1] + t[0,2,-2]$.
(this could also be solved by finding two points on the line. One point $P=(1,1,1)$ was given.
b) We know $(a,b,c) = (0,2,-2)$ from a) so that $0 \cdot x + 2y - 2z = d$ is the equation of the plane. Plugging in a point like P gives the constant so that $2y - 2z = 0$.

Problem 5) (10 points)

a) (5 points) What is the distance between the point $B=(1,2,3)$ to the plane containing $A=(2,2,3)$ and normal vector $\vec{n} = [1,1,1]$.



b) (5 points) What is the distance between the point $B=(1,2,3)$ and the line $\vec{r}(t) = [2+t, 2+t, 3+t]$ that contains $A=(2,2,3)$ and the vector $\vec{n} = [1,1,1]$?

Solution:

These are routine problem: both use $\vec{AB} = [-1,0,0]$.
a) $|\vec{AB} \cdot \vec{n}|/|\vec{n}| = 1/\sqrt{3}$.
b) $|\vec{AB} \times \vec{n}|/|\vec{n}| = \sqrt{2}/\sqrt{3}$.

Problem 6) (10 points)

a) (6 points) Find the arc length of the **"Trust me!"** curve

$$\vec{r}(t) = \begin{bmatrix} e^t \\ 2\sqrt{2}te^{t/2} - 4\sqrt{2}e^{t/2} \\ \frac{t^3}{3} \end{bmatrix}$$

from $0 \leq t \leq 1$. You have to **trust me** that the integral works out nicely!

b) (4 points) Find the curvature of $\vec{r}(t)$ at $t = 0$.

What AI thought about "Trust me!". We can only add: The guy is right, but there is a catch.

Solution:

a) $\vec{r}'(t) = [e^t, \sqrt{2}te^{t/2}, t^2]$ so that $|\vec{r}'(t) = \sqrt{e^{2t} + 2te^t + t^4} = \sqrt{(e^t + t^2)^2} = e^t + t^2$. The arc length is $e - 2/3$.
b) Compute $\vec{r}''(t) = [e^t, \sqrt{2}e^{t/2} + \sqrt{2}te^{t/2}/2, 2t]$. Now determine $\vec{r}'(0) = [1, 0, 0], \vec{r}''(0) = [1, \sqrt{2}, 0]$. Their cross product has length $\sqrt{2}$. The curvature is $\sqrt{2}/1^3 = \sqrt{2}$.

Problem 7) (10 points)

Solve the following gorgeous **jerk=helix problem**: You know that the jerk of an unknown curve satisfies

$$\vec{r}\,'''(t) = \begin{bmatrix} t \\ \cos(t) \\ \sin(t) \end{bmatrix} .$$

You also know $\vec{r}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{r}\,'(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ as well as $\vec{r}\,''(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. What is $\vec{r}(t)$?

What AI suggested about "jerk in mathematics". Do you get it? ($\vec{v}, \vec{a}, \vec{j}$).

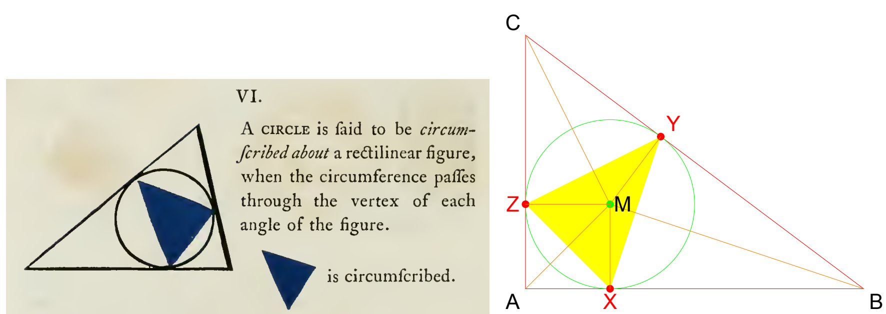
Solution:

Integrate three times and adjust the constant in each step. $r(t) = [t^4/4, \sin(t) + t, \cos(t) + t 2/2 - 1]$.

Problem 8) (10 points)

In a homework you have investigated in inscribed circle of the **3-4-5 triangle** A, B, C. The circle touches the triangle in the points X,Y,Z. Find the area of this triangle X, Y, Z using the cross product area formula.

Hint: We can tell you that $A = (0, 0, 0), B = (4, 0, 0)$ and $C = (0, 3, 0)$ and $Y = (8, 9, 0)/5$ but we do not give you the coordinates of the points X and Z. You should remember this from the homework!



Left: a diagram from Oliver Byrne's gorgeous Euclid interpretation. To the right: the 3-4-5 triangle ABC and the triangle XYZ in the inscribed circle.

Solution:

The remember the radius was 1 from the homework. If you did not remember that (especially with the provided picture which indicates the radius must be close to 1), this is an indication that you have not engaged enough in the homework!
We have $X = (1, 0, 0), Y = (0, 1, 0)$ so that $\vec{XZ} = [-1/5, 9/5, 0], \vec{XY} = [-1, 1, 0]$. The area is $|\vec{XZ} \times \vec{XY}|/2$ which is $6/5$.

Problem 9) (10 points) No justifications are needed.

Oliver recently filmed some sailing boats on the Mystic lakes. He felt the urge to build a mathematical model.

a) (2 points) The body of the boat is part of an ellipsoid $x^2/4 + y^2/2 + z^2 = 1$, parametrized using spherical coordinates:

$\vec{r}(\theta, \phi) = [$ ______ , ______ , ______ $]$

with $0 \leq \theta \leq 2\pi$ and $\pi/2 \leq \phi \leq \pi$.

b) (2 points) The sail is written as an elliptic cone $x^2 + 100y^2 = 3 - z$ is parametrized using coordinates x, y as

$\vec{r}(x, y) = [$ ______ , ______ , ______ $]$

c) (2 points) The sail mast $x^2 + y^2 = 1/25$ is a cylinder parametrized as

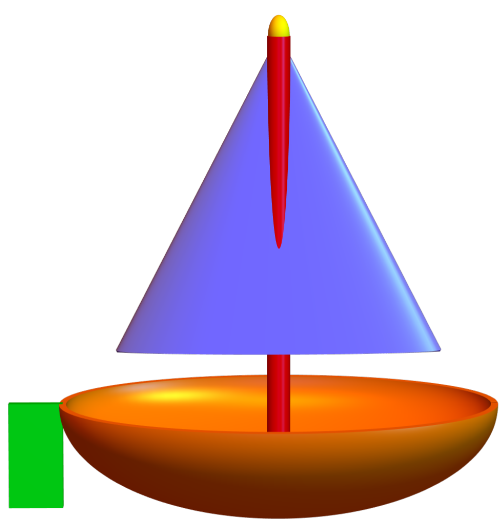
$\vec{r}(\theta, z) = [$ ______ , ______ , ______ $]$

d) (2 points) The top of the mast is a sphere of radius $1/5$ centered at $(0, 0, 7/2)$.

$\vec{r}(\theta, \phi) = [$ ______ , ______ , ______ $]$

e) (2 points) Finally, the rudder is part of the plane $x = 0$. Parametrize it using parameters s, t in the form $\vec{P} + s\vec{v} + t\vec{w}$ for suitable vectors $\vec{v}$ and $\vec{w}$. You write out the coordinates as

$\vec{r}(s, t) = [$ ______ , ______ , ______ $]$



Solution:

a) $[2\sin(\phi)\cos(\theta), \sqrt{2}\sin(\phi)\sin(\theta), \cos(\phi)$.
b) $[x, y, 3 - x^2 - 100y^2]$
c) $[\cos(\theta)/5, \sin(\theta)/5, z]$
d) $[(1/5)\sin(\phi)\cos(\theta), (1/5)\sin(\phi)\sin(\theta), (1/5)\cos(\phi) + 7/5]$
e) $[0, s, t]$.