7/24/2025 SECOND HOURLY **Maths 21a, O.Knill, Summer 2025**

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) **T** F — If the curve $\vec{r}(t)$ is contained in the level surface $x^2 + y^4 x^2 + z^2 = 10$, then $\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$.

2) T **F** — If $f(x, y)$ satisfies the PDE $f_x + f_y = 0$, then f has only critical points with $D < 0$.

Solution:

It follows that $D = 0$.

3) **T** F — Assume that $f(x, y)$ is a solution of the PDE $f_{xx} = -f_{yy}$, and $(0, 0)$ is a critical point of f at which D is not zero, then $(0, 0)$ is a saddle point of f.

Solution:

$D = -f_{xx}^2 - f_{xy}^2 < 0$.

4) **T** F — $-|\nabla f(0, 0)|$ is the minimum among all directional derivative $D_{\vec{v}} f(0, 0)$ that can be computed at $(0, 0)$.

Solution:

Angles dance upwards, devils dance downwards. The maximal slope indeed is the length of the gradient.

5) T **F** — The famous Bell curve function $f(x) = e^{-x^2}$ has an anti-derivative in terms of $\sin, \cos, \exp, \log$ and power x^n functions.

Solution:

As mentioned.

6) T **F** — If a function $f(x, y)$ satisfies $f_{xx}(0, 0) > 0, f_{yy}(0, 0) > 0$, then f has a minimum at $(0, 0)$.

Solution:

Take $f(x, y) = 10000xy - x^2 - y^2$ is a counter example. We discussed this in class.

7) T **F** $(0,0)$ is a critical point of the function $f(x,y) = x^8 - y^8$. As it is a saddle D is non-zero.

Solution:

The function is non-negative but 0 at the origin.

8) **T** F There are smooth functions $f(x,y)$ and $g(x,y)$, such that $(0,0)$ is minimum of f on the curve $g(x,y) = 0$ but for which the Lagrange equations fail.

Solution:

We have seen an example

9) **T** F The chain rule tells that $\frac{d}{dt} g(\vec{v}(t)) = \nabla g(\vec{v}(t)) \cdot \vec{v}\,'(t)$ if g is a function and $\vec{v}(t)$ is a curve.

Solution:

We just used other letters

10) **T** F If R the unit square $[0,1] \times [0,1]$ then $0 < \iint_R \sin^2(x^2)\cos(y^2)\, dA < 1$.

Solution:

It is the volume of a solid contained in the unit box.

11) T **F** If $f(x,y,z) = x^2 + y^8 y - z^2 = 1$ defines z as a function $z = g(x,y)$ near the point $(1,1,1)$ then by implicit differentiation, $g_y(1,1) = -f_z(1,1,1)/f_y(1,1,1)$.

Solution:

Wrong order: it is f_y/f_z.

12) **T** F The following identity holds $\int_0^1 \int_0^{x^x} x^{-x} dydx = 1$.

Solution:

Just integrate

13) T **F** The following identity holds $\int_0^1 \int_0^1 x^{-x} \, dxdy = 1$.

Solution:

x^{-x} is smaller than 1

14) **T** F The function $f(x,y) = 4x^2y^2/(y^2+x^2)$ is an example, where Clairaut fails.

Solution:

We have seen this in class.

15) **T** F Let $f(x,y) = x^7y^2$. At every point (x,y) there is a direction $\vec{v}$ for which $D_{\vec{v}}f(x,y) = 0$.

Solution:

Since $D_v f(x,y) = -D_{-v}f(x,y)$, a directional derivative once positive in one direction, is negative in the opposite direction.

16) **T** F For $\vec{u} = [1,0]$ and $\vec{v} = [0,1]$ the directional derivatives satisfy $D_{\vec{v}}D_{\vec{u}}f = D_{\vec{u}}D_{\vec{v}}f$.

Solution:

This is Clairaut.

17) T **F** If $f(x,y)$ has two local minima in the plane then f must have a local maximum on the plane.

Solution:

A counter example is $-x^2 exp(-x^2 - y^2)$.

18) | T | **F** | It is possible that the discriminant of a smooth function $f(x,y)$ satisfies $D > 0$ and $f_{xx} = 0$ at a critical point.

Solution:

This is not possible.

19) | **T** | F | The function $f(x,y) = \cos(e^y)x^2 \exp(\sin(y^4))$ satisfies the partial differential equation $f_{xxyyyxyy} = 0$.

Solution:

By Clairots theorem, we can have all three x derivatives at the beginning.

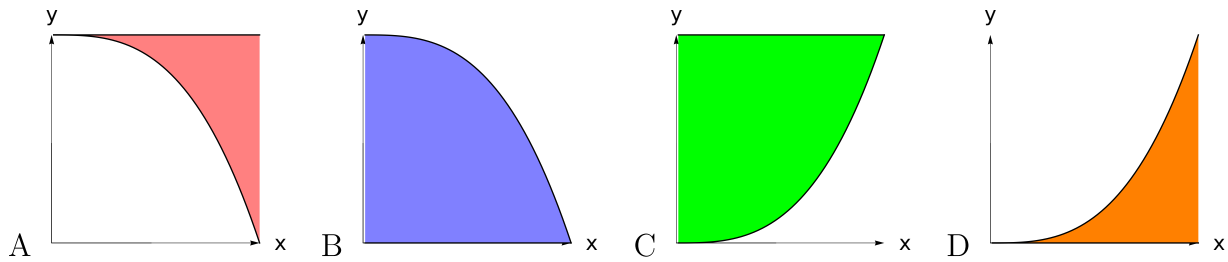
20) | **T** | F | If for all unit vectors $\vec{v}$ in the plane, the function $g(x,y) = D_{\vec{v}} f(x,y)$ is zero at $(0,0)$, then $(0,0)$ is a critical point.

Solution:

It implies the gradient is zero.

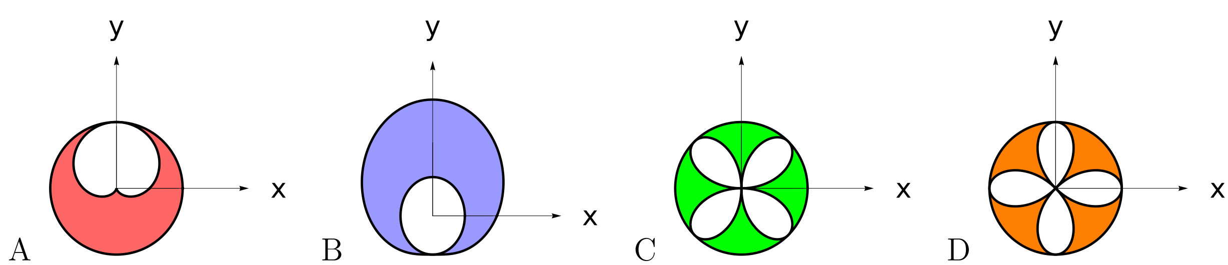
Problem 2) (10 points) No justifications are needed in this problem.

a) (4 points) Please match the regions with the moment of inertia formulas. A-D are used exactly once.



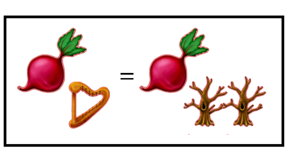
Enter A-D	Moment of inertia integral
	$\int_0^1 \int_0^{y^{1/3}} (x^2+y^2)\, dxdy$
	$\int_0^1 \int_0^{x^3} (x^2+y^2)\, dydx$
	$\int_0^1 \int_0^{1-x^3} (x^2+y^2)\, dydx$
	$\int_0^1 \int_{(1-y)^{1/3}}^1 (x^2+y^2)\, dxdy$

b) (4 points) Please match **polar regions** with area integrals. A-D are used exactly once. Pictures do not necessarily use the same scale. These examples emerged while designing ear rings.

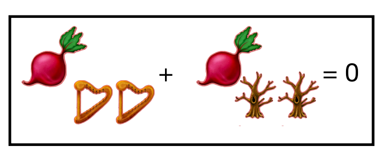


Enter A-D	Moment of inertia integral
	$\int_0^{2\pi} \int_{2\cos(2t)}^2 r^3 dr d\theta$
	$\int_0^{2\pi} \int_{2\sin(2t)}^2 r^3 dr d\theta$
	$\int_0^{2\pi} \int_1^{2+\sin(t)} r^3 dr d\theta$
	$\int_0^{2\pi} \int_{1+\sin(t)}^2 r^3 dr d\theta$

c) (2 points) Identify the following two partial differential equations with name. We used as symbols some new emojis introduced in 2025:



represents the PDE



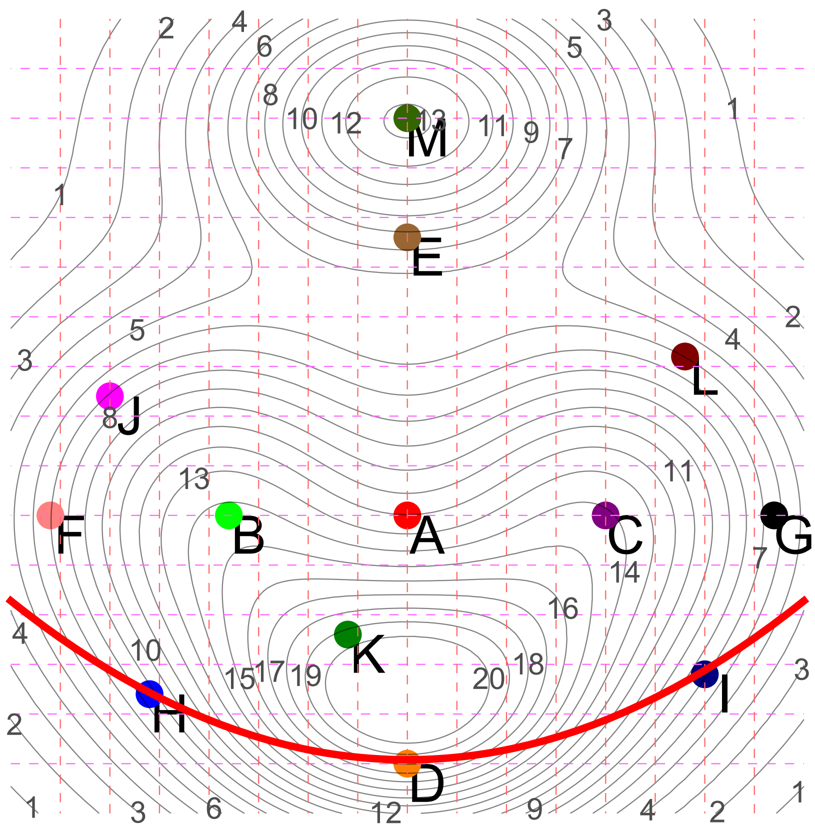
represents the PDE

Solution:

a) CDBA
b) DCBA
c) Heat, Laplace

Problem 3) (10 points) No justifications are needed in this problem.

You see a contour map of a smooth function $f(x, y)$. The thick curve through H, D, I is a level curve $g(x, y) = 0$ that is also parametrized as $\vec{r}(t)$. Points can occur several times.



	Enter from $\{A - M\}$
A point that is a local maximum of f	
A point that is a maximum of f on the constraint $g(x, y) = 0$	
Three points that are non-critical points of f, where $f_x = 0$	
A point, where $|\nabla f|$ is a local maximum	
A point, where $|\nabla f|$ is a local minimum	
A point on the curve, where $\frac{d}{dt} f(\vec{r}(t)) = 0$	
A point, where $f_x = 0, f_{xx} > 0$.	
A point, where ∇f is a negative multiple of $\vec{i} = [1, 0]$	
Two points, where ∇f is a positive multiple of $\vec{j} = [0, 1]$	
A point, where $f_x > 0$ and $f_y > 0$.	

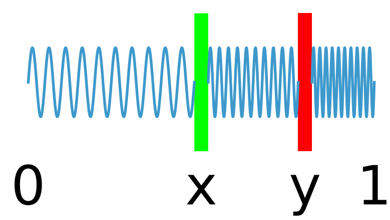
Solution:

M,D, AED, D,M,D,A,G,DE,H

Problem 4) (10 points)

The kinetic energy $f(x, y)$ of two **mass points** with position $x \in [0, 1]$ and $y \in [0, 1]$ is

$$f(x, y) = x^2 + 2(x - y)^2 + 3(1 - y)^2$$



The coupling strength of the spring between the two bodies is twice the coupling strength of the left spring and the right spring is three times as strong.

a) (8 points) Find the minimal energy configuration. Use the second derivative test in order to classify all critical points of f.

b) (2 points) The function f has a global maximum $(0, 0)$ on the domain $\{(x, y), 0 \leq x \leq y \leq 1\}$. You probably did not find this point in a). Why not?

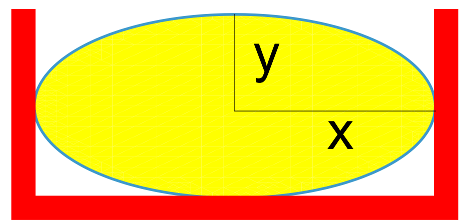
Solution:

We have to solve $5x - 4y = 0, 10y - 4x - 6 = 0$. The only solution is $(x, y) = (6/11, 9/11)$. $D = 60 - 16 = 44 > 0$ and $f_{xx} = 6 > 0$ so that we have a minimum

x	y	D	f_{xx}	Type
6/11	9/11	44	6	min

Problem 5) (10 points)

There is the saying **"Don't put all your eggs in one basket"**. Well, we have only one egg and only one basket. The egg of elliptic shape has the area $f(x, y) = \pi xy$, as we know this was the "hardest problem in geometry". The basket is open has a fixed material $g(x, y) = 2x + 4y = 8$. Use the Lagrange method to find the geometric parameters (x, y) for which the area of the egg is maximized.



Solution:

The Lagrange equations are

$$\begin{aligned} \pi x &= \lambda 2 \\ \pi y &= \lambda 4 \\ 2x + 4y &= 8 \end{aligned}$$

It gives $x = 2, y = 1$.

Problem 6) (10 points)

a) (2 points) Estimate $2.1 \cdot 1.001^{0.9999}$ by linear approximation using the function

$$f(x, y, z) = zx^y = ze^{y \log(x)}$$

and estimate near the point $(1, 1, 2)$.

b) (2 points) What is the directional derivative $D_{\vec{v}} f$ for $\vec{v} = [3, 4, 12]/13$ at the same point $(1, 1, 2)$.

c) (2 points) In which direction $\vec{v}$ do you want to go at $(1, 1, 2)$ to increase f fastest?

d) (2 points) Find the tangent plane $ax + by + cz = d$ to the surface $f(x, y, z) = 2$ at $(1, 1, 2)$.

e) (2 points) Consider the curve $\vec{r}(t) = [1+3t, 1+\sin(2t), 2+\sin(t)]$. It passes at at time $t = 0$ through $(1, 1, 2)$. Find $\frac{d}{dt} f(\vec{r}(t))$ at $t = 0$ using a theorem we have seen.

Solution:

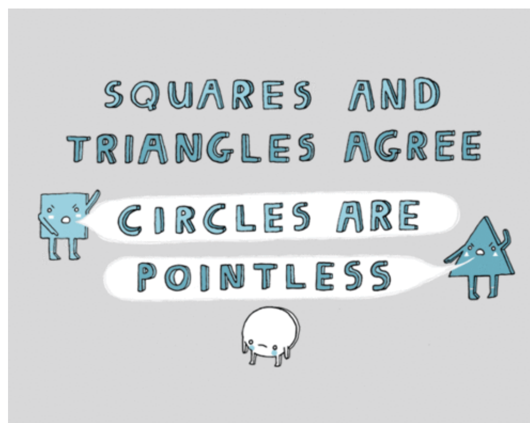
The gradient is $[yzx^{y-1}, zx^y \log(x), x^y]$. Evaluated at the point, this is $[2, 0, 1]$. This is the key to solve all parts!
a) $2 + 2 * 0.001 + 0 * (-0.0001) + 1 * 0.1 = 2.102$.
b) $D_{\vec{v}} f = [2, 0, 1] \cdot [3, 4, 12]/13 = 18/13$.
c) The direction is $[2, 0, 1]/\sqrt{5}$.
d) $2x + 0y + z = d$. Plug in the point to see $d = 4$.
e) $[2, 0, 1] \cdot [3, 2, 1] = 7$.

Problem 7) (10 points)

a) (6 points) Find the surface area of the surface

$$\vec{r}(u, v) = \begin{bmatrix} 3uv \\ 5u^3 \\ 4uv \end{bmatrix}$$

parametrized on the **triangle** domain $R = \{0 \le u \le 1, 0 \le v \le u\}$.



Note that even if the outcome should be clear to you, please give details, as always.

b) (4 points) We parametrize the flat disk $x^2 + y^2 \le 1, z = 0$ enclosed by a **circle** in the xy plane with

$$\vec{r}(r, \theta) = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \\ 0 \end{bmatrix} .$$

Write down the **surface area integral** for the **square** domain $R = \{(r, \theta), 0 \le r \le 1, 0 \le \theta \le 2\pi\}$. Then evaluate the area.

Solution:

a) Use $\iint_R |r_u \times r_v| \, dvdu$ for the surface area.
$\vec{r}_u = [3v, 15u^2, 4v]$
$\vec{r}_v = [3u, 0, 4u]$
$r_u \times r_v = [60u^3, 0, -45u^3]$ has length $75u^3$. (factoring 15 out helped here!) Integrate $\int_0^1 \int_0^u 75u^3 \, dvdu = \int_0^1 75u^4 \, du = 75/5 = 15$.
b) $\vec{r}_r = [\cos(\theta), \sin(\theta), 0], \vec{r}_\theta = [-r\sin(\theta), r\cos(\theta), 0]$. The cross product is $[0, 0, r]$ with magnitude r. So, $\int_0^{2\pi} \int_0^1 r \, drd\theta = \pi$.

Problem 8) (10 points)

We aim to find the volume integral

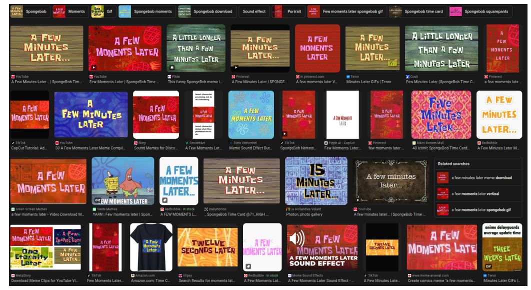
$$\int_0^1 \int_0^{\arccos(y)} \int_0^{e^{\sin(x)}} 1 \; dz \; dx \; dy \; .$$

As triple integrals should not occur yet in the second midterm, we do the inner integral for you.

"A few minutes later ..." (*)

$$\int_0^1 \int_0^{\arccos(y)} e^{\sin(x)} \; dx \; dy \; .$$

Now its your turn to compute this integral.



(*) If you also watch too much Tik-Tok or youtube shorts, you know the term. The meme usually appears spoken in a French accent.

Solution:

Change the order of integration. To find the bound, draw the graph of the cos function! $\int_0^{\pi/2} \int_0^{\cos(x)} e^{\sin(x)} \; dydx = \int_0^{\pi/2} e^{\sin(x)} \cos(x) \; dx = e^{\sin(x)}|_0^{\pi/2} = e - 1$.

Problem 9) (10 points)

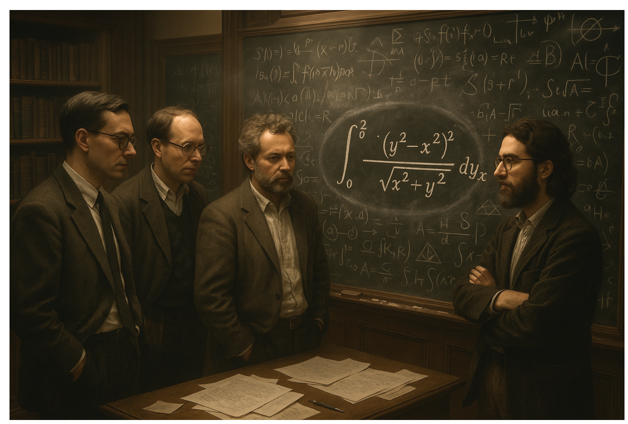
When trying to integrate

$$f(x,y) = \frac{(y^2 - x^2)^2}{\sqrt{x^2 + y^2}}$$

over a pizza slice $\{x^2 + y^2 \leq 4, x < 0, y > 0\}$, we are led to the nasty integral

$$\int_{-2}^0 \int_0^{\sqrt{4-x^2}} \frac{(y^2 - x^2)^2}{\sqrt{x^2 + y^2}} \; dydx \; .$$

Help out to compute this mess.



Picture AI generated. Did not quite succeed to paint the formula correctly onto the board.

Solution:

Use polar coordinates using $x = r\cos(\theta), y = r\sin(\theta)$ and $x^2 + y^2 = r^2$ and $x^2 - y^2 = r^2\cos(2\theta)$. The pizza is a quarter disk in the second quadrant.
$\int_{\pi/2}^{\pi} \int_0^2 \frac{r^4 \cos^2(2\theta)}{\sqrt{r^2}} r dr d\theta = 8\pi/5$. (Use the double angle formula to integrate $\cos^2(2\theta) = (1 + \cos(4\theta))/2$ from $\pi/2$ to π).